The In-betweens

Not-whole numbers and their various forms

Booklet 4 (of 5)

In this booklet:

- **Multiplying fractions – making sense of it**
- **Multiplying fractions – how to do it**
- **Dividing fractions – making sense of it**
- **Dividing fractions – how to do it**
- **Multiplying mixed numbers**
- **Dividing mixed numbers**

Student Name: ________________________________

Teacher Name: ________________________________

Class: ________________________________

Commencement date: ________________________________

Smooth Mathematics

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The In-Betweens
Not-whole numbers and their forms

Booklet 4 (of 5)

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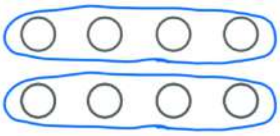
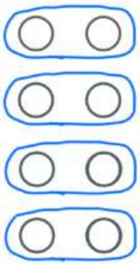
22. Operations involving fractions – part 2

22.1 Multiplication involving fractions – what makes sense?

The number of objects in *two* groups of 4 (or "two 4s") is equal to the number of objects in *four* groups of 2 (or "four 2s"), as shown in the diagrams immediately below.





"Groups of" can be thought about as *multiplication* and so:

$$2 \times 4 = 4 \times 2 = 8$$

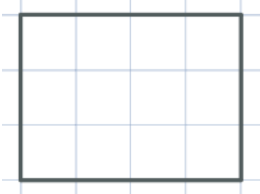
Perhaps this is not new to you. ☺

What about $\frac{1}{3}$ groups of $\frac{1}{4}$ (of some whole)?

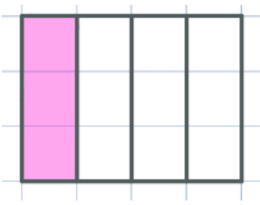
It might be better, grammatically, to ask,

What about $\frac{1}{3}$ *of a group* of $\frac{1}{4}$ (of some whole)?

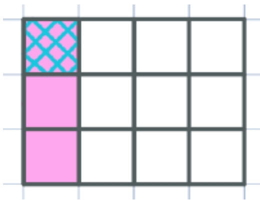
Suppose the whole, in this case, is a rectangle 3 units tall and 4 units wide, as shown immediately below.



$\frac{1}{4}$ of the rectangle can be illustrated by the pink section, shown immediately below:



$\frac{1}{3}$ of the $\frac{1}{4}$ of the rectangle can be illustrated by the cross-hatched section, shown immediately below:

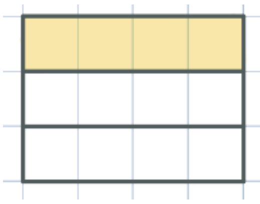


Based on this example, it makes sense that:

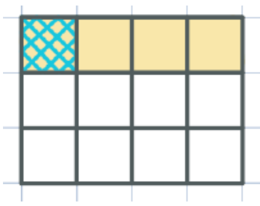
$$\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$$

What about the *reverse*, $\frac{1}{4}$ *of a group* of $\frac{1}{3}$ (of some whole)?

Suppose the whole, in this case, is again a rectangle 3 units tall and 4 units wide.
$\frac{1}{3}$ of the rectangle can be illustrated by the yellow section, shown immediately below:



$\frac{1}{4}$ of the $\frac{1}{3}$ of the rectangle can be illustrated by the cross-hatched section, shown immediately below:



Based on this example, it makes sense that:

$$\frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$$

and so:

$$\frac{1}{3} \times \frac{1}{4} = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$$

At least in this example, multiplication involving two fractions has a similarity to multiplying whole numbers, i.e. $4 \times 3 = 12$ and $3 \times 4 = 12$. ☺

Question 1

Draw, colour and cross-hatch one rectangle to show that it makes sense that

$$\frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$$

Question 2

Draw, colour and cross-hatch two rectangles to show that it makes sense that

$$\frac{2}{3} \times \frac{1}{7} = \frac{1}{7} \times \frac{2}{3} = \frac{2}{21}$$

Question 3

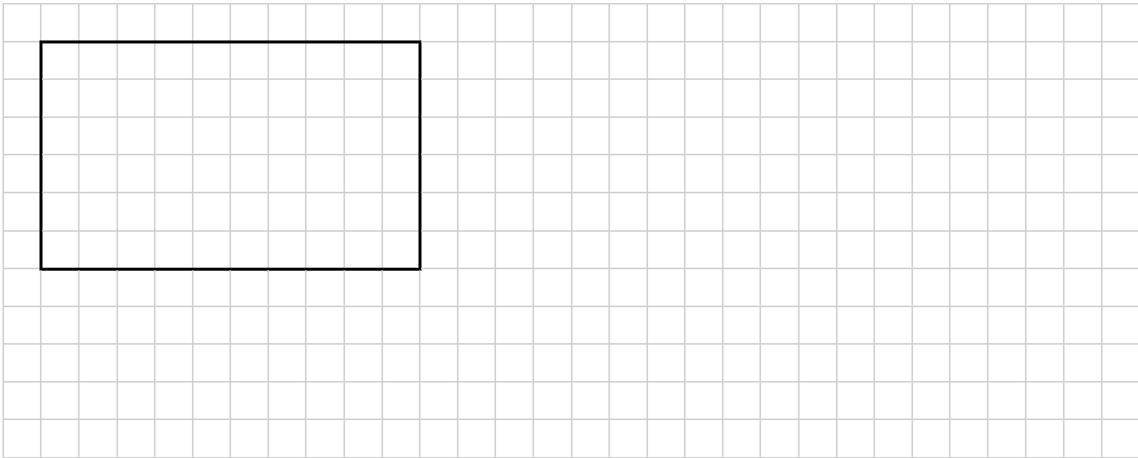
Draw, colour and cross-hatch two rectangles to show that it makes sense that

$$\frac{3}{7} \times \frac{2}{5} = \frac{2}{5} \times \frac{3}{7} = \frac{6}{35}$$

Question 4

Colour and cross-hatch the following rectangle to show that it makes sense that

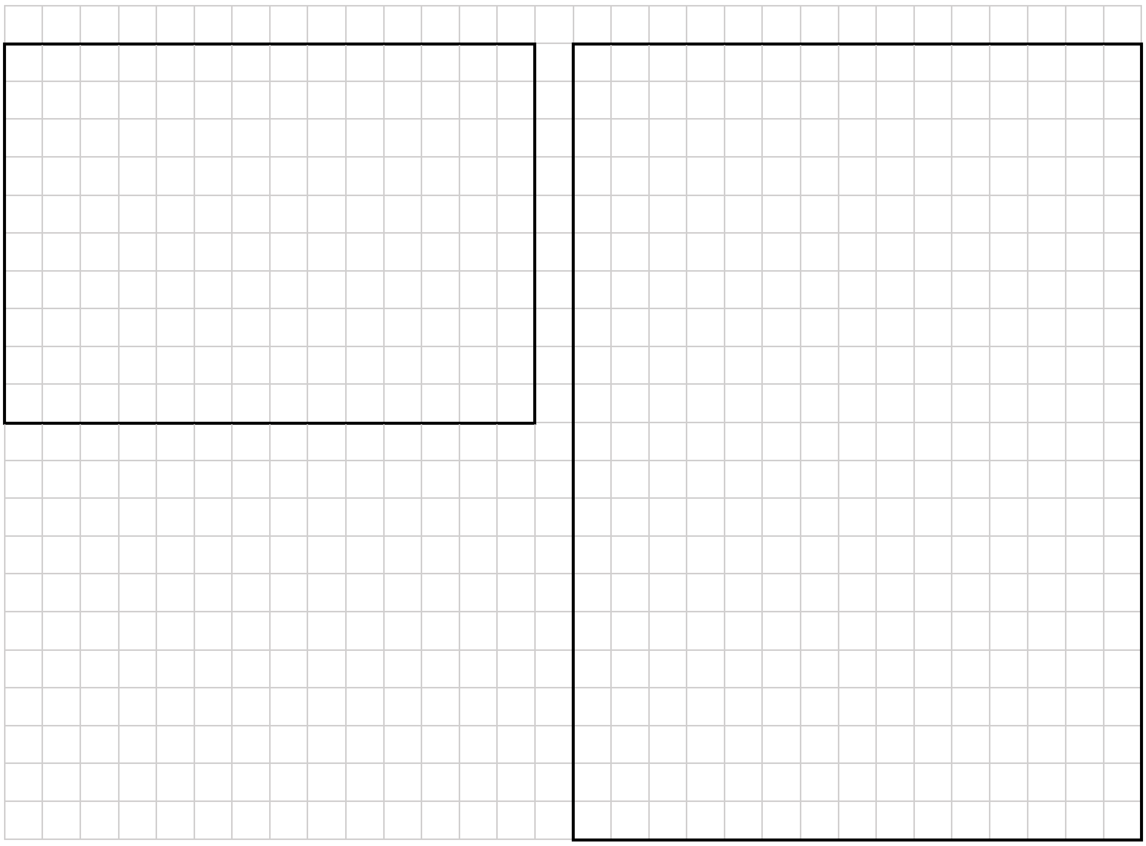
$$\frac{2}{3} \times \frac{2}{5} = \frac{4}{15}$$



Question 5

Colour and cross-hatch both of the following rectangles to show that it makes sense that

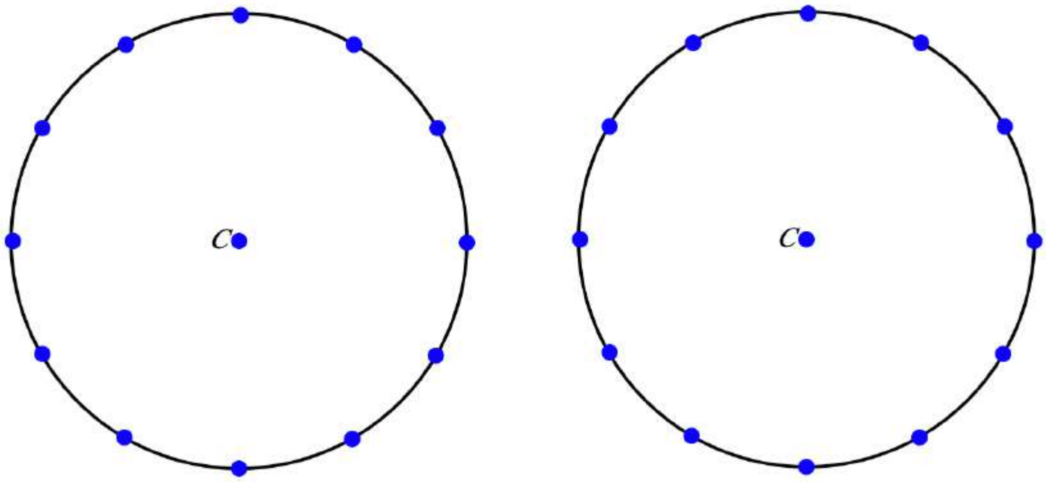
$$\frac{3}{5} \times \frac{2}{7} = \frac{6}{35}$$



Question 6

Use this pair of "clock-circles" to illustrate that

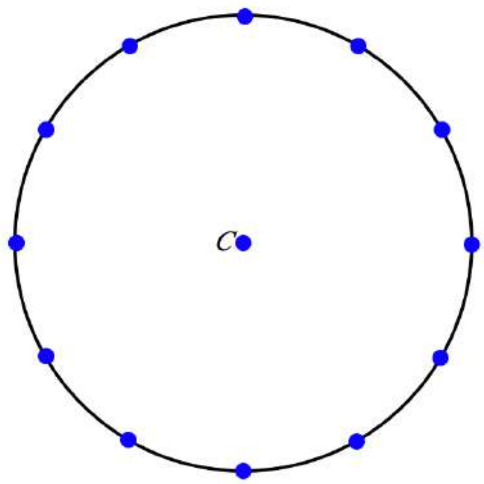
$$\frac{1}{3}\times\frac{1}{4}=\frac{1}{4}\times\frac{1}{3}=\frac{1}{12}$$



Question 7

Use this "clock-circle" to illustrate that

$$\frac{2}{7}\times\frac{7}{12}=\frac{1}{6}$$



22.2 Another way to think about multiplication involving fractions

In this section you will think about fractions in a more *number-y* way.

We have previously established that three groups of 4 can be written as 3×4.

But three groups of 4 can also be thought of as $4 + 4 + 4$.

And so, $3 \times 4 = 4 + 4 + 4 = 12$.

Here, we multiply, by adding three 4s together, *not* four dots or dogs, but 4s.
The result is the number 12, not twelve 1-unit squares in a rectangle or dots or …

Numbers do not have to be the count of some object. A number can just be a number. ☺

But what about $\frac{1}{5} \times \frac{10}{11}$?

The way of thinking described immediately above might seem hard to apply in this case.
So, we will have to create a *different* way of thinking.

Think about $\frac{10}{11}$ as ***ten*** $\frac{1}{11}$s.
The ***ten*** is a count of how many $\frac{1}{11}$s we have.
The $\frac{1}{11}$ can be thought of as just a "thing" and so we have ***ten*** of these things.

It could have been ten dots or ten dogs, but in this case it is ten $\frac{1}{11}$s.

Now, $\frac{1}{5}$ of ***ten*** is two and so ultimately, we have ***two*** "things", ***two*** one-elevenths. Or, more simply, $\frac{2}{11}$.

So:

$$\frac{1}{5} \times \frac{10}{11} = \frac{2}{11}$$

Got it?
What would $\frac{1}{4} \times \frac{12}{13}$ equal?

Well, $\frac{1}{4}$ of ***twelve*** things is ***three*** things and so $\frac{1}{4} \times \frac{12}{13} = \frac{3}{13}$

Ok, what would $\frac{1}{3} \times \frac{3}{4}$ equal?
Well, $\frac{1}{3}$ of ***three*** things is ***one*** thing and so $\frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$

Nice, isn't it. 🙂

However, …

How would you reason, without a diagram would $\frac{1}{3} \times \frac{1}{4}$ equals? 😳😬

We can reason, without a diagram what $\frac{1}{3} \times \frac{1}{4}$ equals in the following way.

$\frac{1}{3}$ of **one** thing is **one-third** of a thing and so,

$$\frac{1}{3} \times \frac{1}{4} = \frac{\frac{1}{3}}{4}$$

Oh my, that looks a little weird, a fraction with a fraction as the numerator! 🙂
It turns out it is just fine, if not a little scary, but there is a much cleaner way to think about this.

We want one-third of something. So instead of having *one* thing, it would have been so nice to have had *three* things.
Mmm, 🤔 …, equivalent (equal) fractions …, could that idea be helpful?

We know that $\frac{1}{4} = \frac{3}{12}$ and so $\frac{1}{4}$ can be thought of as three things! It is three $\frac{1}{12}$s, or $\frac{3}{12}$.

Now, $\frac{1}{3}$ of three things is *one thing* and so,

$$\frac{1}{3} \times \frac{1}{4} = \frac{1}{3} \times \frac{3}{12} = \frac{1}{12}$$

Ok, what would $\frac{1}{4} \times \frac{1}{9}$ equal?

Can you think how to proceed?

It would be so nice to have four things, not one thing.

We know that $\frac{1}{9} = \frac{4}{36}$ and so $\frac{1}{4} \times \frac{1}{9} = \frac{1}{4} \times \frac{4}{36}$

Now, $\frac{1}{4}$ of *four* things is *one thing* and so $\frac{1}{4} \times \frac{1}{9} = \frac{1}{4} \times \frac{4}{36} = \frac{1}{36}$

Have you noticed anything that might "shortcut" this process?

Did you pick up that it seems that

$$\frac{1}{b} \times \frac{1}{d} = \frac{1}{b \times d}$$

Indeed, this is true.

We know that $\frac{1}{d} = \frac{b}{b \times d}$ and so $\frac{1}{b} \times \frac{1}{d} = \frac{1}{b} \times \frac{b}{b \times d}$

Now, $\frac{1}{b}$ of b things is *one thing* and so $\frac{1}{b} \times \frac{1}{d} = \frac{1}{b} \times \frac{b}{b \times d} = \frac{1}{b \times d}$

Therefore,

$$\frac{1}{b} \times \frac{1}{d} = \frac{1}{b \times d}$$

☺

Question 1

Use the style of reasoning shared in this section to reason why:

a) $\frac{1}{5} \times \frac{5}{12} = \frac{1}{12}$ b) $\frac{1}{8} \times \frac{8}{11} = \frac{1}{11}$ c) $\frac{1}{c} \times \frac{c}{8} = \frac{1}{8}$

Question 2

Use the style of reasoning shared in this section to reason why:

a) $\frac{1}{6} \times \frac{18}{31} = \frac{3}{31}$ b) $\frac{1}{7} \times \frac{28}{29} = \frac{4}{29}$ c) $\frac{1}{9} \times \frac{72}{7} = \frac{8}{7}$

Question 3

Use the style of reasoning shared in this section to reason why:

a) $\frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$ b) $\frac{1}{3} \times \frac{1}{9} = \frac{1}{27}$ c) $\frac{1}{20} \times \frac{1}{20} = \frac{1}{400}$

Question 4

In this section it was shown, in a *numbery/pronumeral* way, that

$$\frac{1}{b} \times \frac{1}{d} = \frac{1}{b \times d}\,.$$

Do your best to explain, in a *pronumeral* way, why

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}\,.$$

(*a*, *b*, *c* and *d* take the place of any whole number.)

22.3 Multiplication involving fractions – the algorithm

When multiplying two fractions, it is true for all cases, that the correct result can be calculated by:

- multiplying the two numerators to give the numerator of the resulting fraction and then,
- multiplying the two denominators to give the denominator of the resulting fraction.

Or expressed in another way,

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

Example 1.

Calculate $\frac{6}{7} \times \frac{5}{11}$

Give the result in simplest form.

0

$\frac{6}{7} \times \frac{5}{11}$

$= \frac{6 \times 5}{7 \times 11}$

$= \frac{30}{77}$ (Which is in simplest form.)

Example 2.

Calculate $\frac{3}{5} \times \frac{2}{9}$

Give the result in simplest form.

0

$\frac{3}{5} \times \frac{2}{9}$

$= \frac{3 \times 2}{5 \times 9}$

$= \frac{6}{45}$ (Which is NOT in simplest form.)

$= \frac{2}{15}$ (Which is in simplest form.)

When multiplying two fractions, *before we start*, we can recognise whether or not the result will be in simplest form.

But how?

If *any* of the numerators share a factor with *any* of the denominators, the result will not be in simplest form after multiplying.

Thus, it can be easier to *simplify* before multiplying, as shown in the following examples.

Example 3.

Calculate $\frac{3}{7} \times \frac{5}{9}$

Give the result in simplest form.

0

$$\frac{3}{7} \times \frac{5}{9}$$

$$= \frac{3 \times 5}{7 \times 9} \qquad \frac{\overset{1}{\cancel{3}}}{\cancel{3} \times 3}$$

$$= \frac{5}{7 \times 3}$$

$$= \frac{5}{21}$$

(Which is in simplest form.)

Example 4.

Calculate $\frac{3}{8} \times \frac{6}{5}$

Give the result in simplest form.

0

$$\frac{3}{8} \times \frac{6}{5}$$

$$= \frac{3 \times 6}{8 \times 5} \qquad \frac{\overset{1}{\cancel{2}} \times 3}{\cancel{2} \times 2 \times 2}$$

$$= \frac{3 \times 3}{4 \times 5}$$

$$= \frac{9}{20}$$

(Which is in simplest form.)

Question 1

Calculate each of the following.

a) $\frac{1}{4} \times \frac{7}{9}$

b) $\frac{1}{7} \times \frac{1}{9}$

c) $\frac{3}{7} \times \frac{4}{11}$

d) $\frac{8}{9} \times \frac{7}{9}$

e) $\frac{12}{7} \times \frac{5}{11}$

f) $\frac{6}{7} \times \frac{8}{9}$

Question 2

Calculate each of the following.

a) $\frac{1}{4} \times \frac{4}{9}$

b) $\frac{2}{7} \times \frac{5}{8}$

c) $\frac{2}{15} \times \frac{5}{7}$

d) $\frac{1}{7} \times \frac{14}{15}$

e) $\frac{6}{7} \times \frac{5}{9}$

f) $\frac{12}{5} \times \frac{7}{18}$

Question 3

Calculate each of the following.

a) $\frac{3}{9} \times \frac{4}{5}$

b) $\frac{5}{7} \times \frac{6}{8}$

c) $\frac{5}{6} \times \frac{4}{15}$

d) $\frac{4}{21} \times \frac{7}{10}$

e) $\frac{4}{6} \times \frac{18}{24}$

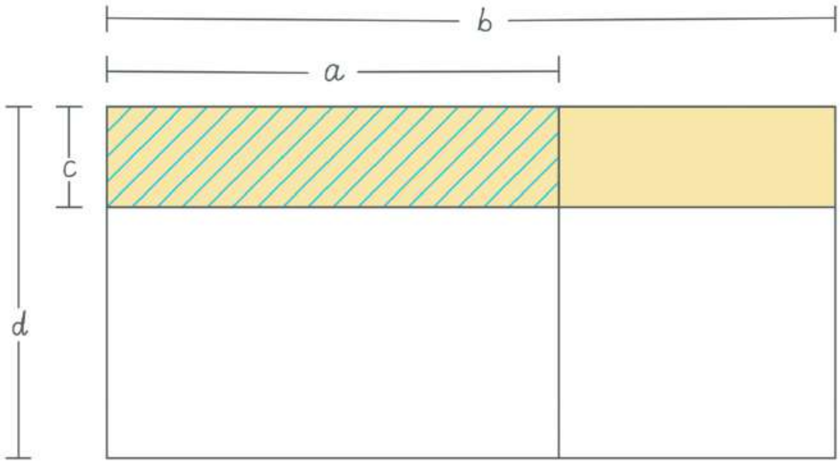
f) $\frac{22}{15} \times \frac{26}{45}$

Question 4

The following diagram depicts a rectangle b units wide and d units tall.

The region shaded yellow is a rectangle b units wide and c units tall.

The region with green stripes is a rectangle a units wide and c units tall.



Do your best to explain, in your own words, how this diagram illustrates the truth of the statement,

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

(You might like to imagine unit squares in the regions of interest.)

22.4 The reciprocal – aka the multiplicative inverse

What is $\frac{1}{5} \times 5$ or $\frac{1}{5} \times \frac{5}{1}$?

1.

What is $\frac{1}{12} \times 12$ or $\frac{1}{12} \times \frac{12}{1}$?

1.

What is $\frac{1}{n} \times \frac{n}{1}$?

1.

What is $\frac{3}{4}$ of $\frac{4}{3}$, or $\frac{3}{4} \times \frac{4}{3}$?

1.

What is $\frac{2}{5} \times \frac{5}{2}$?

1.

What $\frac{5}{3} \times \frac{3}{5}$?

1.

Get the idea? ☺

If the fraction, $\frac{a}{b}$, is transformed into a different fraction by swapping the numerator and denominator, it becomes $\frac{b}{a}$.

$\frac{b}{a}$ is called the *reciprocal* of $\frac{a}{b}$.

$\frac{b}{a}$ is also called the *multiplicative inverse* of $\frac{a}{b}$.

The idea of the multiplicative inverse is important, because when a fraction and its multiplicative inverse are multiplied, the result is 1.

$$\frac{a}{b} \times \frac{b}{a} = \frac{a \times b}{b \times a} = 1$$

1 is an important number where multiplication and division are concerned, because **multiplying any number by 1, or dividing any number by 1, *does not change the value of the number***.

$$a \times 1 = 1 \times a = a$$

$$a \div 1 = a$$

You will see why this is important in the next section.

Question 1

What is the reciprocal (multiplicative inverse) of each number?

a) $\frac{1}{4}$
b) 4
c) 3
d) $\frac{9}{8}$
e) $\frac{3}{7}$
f) $\frac{14}{18}$

Question 2

Each of the following numbers can be multiplied by a number to give a result of 1. What is the number in each case? Write the number as a fraction in simplest form.

a) $5\frac{1}{2}$
b) $7\frac{2}{9}$
c) $8\frac{5}{6}$
d) 0.75
e) 0.2
f) 0.125

Question 3

Each of the following numbers can be multiplied by a number to give a result of 1. What is the number in each case? Write the number as a fraction in simplest form.

a) 2.1
b) 2.5
c) 4.3
d) 1.9
e) 12.4
f) 0.00042

22.5 Division involving fractions – what does it mean?

What does $2 \div \frac{1}{4}$ mean?

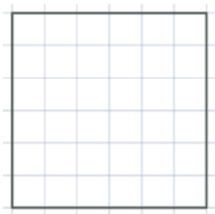
Remember that one way to think about division is as *how many* of *those* are in *that*. To calculate $12 \div 4$, we can ask how many 4s are in 12, and we know that is 3.

So, one way to think about $2 \div \frac{1}{4}$ is to ask yourself

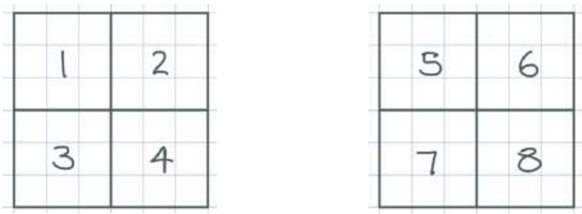
how many $\frac{1}{4}$s (of a whole) are in 2 (of the *same* whole)?

(Clearly more than 1, right?)

Suppose the whole is a square, as shown immediately below.



In two such squares we can see that there are *eight* $\frac{1}{4}$s (of a whole):

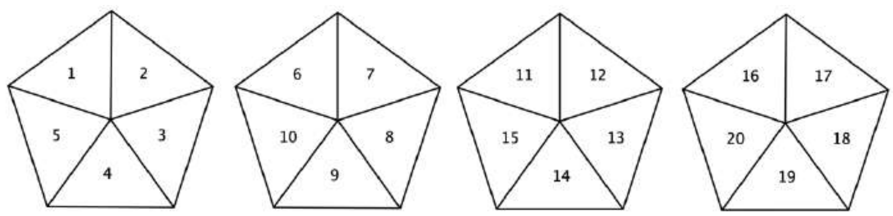


So, it makes sense that $2 \div \frac{1}{4} = 8$.

Alternatively, we could note there are *eight* $\frac{1}{4}$*s in* 2 because:

$$\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{8}{4}=2$$

What is the result of $4 \div \frac{1}{5}$?
I.e., how many $\frac{1}{5}$s (of a whole) are in 4 (of the *same* whole)?

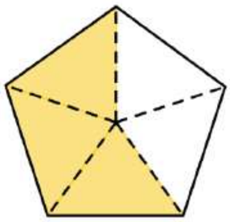


It makes sense that $4 \div \frac{1}{5} = 20$.

Alternatively, we could note there are *twenty* $\frac{1}{5}$*s in 4* because:

$$\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{20}{5}=4$$

What is the result of $\frac{3}{5} \div \frac{1}{5}$?



We can count three $\frac{1}{5}$s in the $\frac{3}{5}$ that is shaded yellow.

So, it makes sense that $\frac{3}{5} \div \frac{1}{5} = 3$.

Alternatively, we could note there are *three* $\frac{1}{5}$s in $\frac{3}{5}$ because:

$$\frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5}$$

What is the result of $\frac{8}{9} \div \frac{2}{9}$?
Well,

$$\frac{2}{9} + \frac{2}{9} + \frac{2}{9} + \frac{2}{9} = \frac{8}{9}$$

So, it makes sense that $\frac{8}{9} \div \frac{2}{9} = 4$.

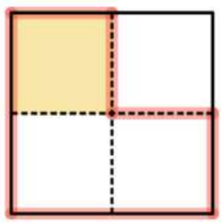
Notice that in all the examples presented so far, *the divisor has been smaller than the dividend*, which will always return a result (quotient) larger than 1.

What if the divisor was *larger* than the dividend?

What is the result of $\frac{1}{4} \div \frac{3}{4}$?

I.e., *how many* (or how much of) $\frac{3}{4}$s (of a whole) is in $\frac{1}{4}$ (of the *same* whole)?

In the diagram immediately below, can you see there is *less than one* $\frac{3}{4}$-piece in a $\frac{1}{4}$-piece.



In fact, $\frac{1}{3}$ *of the* $\frac{3}{4}$*-piece* is the yellow shaded $\frac{1}{4}$-piece. (Think of the $\frac{3}{4}$*-piece* as the new whole.)

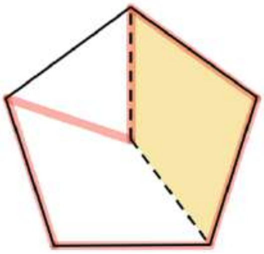
So, it makes sense that $\frac{1}{4} \div \frac{3}{4} = \frac{1}{3}$.

(How many $\frac{3}{4}$s would I need to add up to get $\frac{1}{4}$? 🤔)

What is the result of $\frac{2}{5} \div \frac{4}{5}$?

I.e. how much of $\frac{4}{5}$s (of a whole) is in $\frac{2}{5}$s (of the *same* whole)?

In the diagram below, we can see that there is *less than one* $\frac{4}{5}$-piece in a $\frac{2}{5}$-piece.



In fact, $\frac{1}{2}$ of the $\frac{4}{5}$-piece is the yellow shaded $\frac{2}{5}$-piece. (Think of the $\frac{4}{5}$-piece as the new whole.)

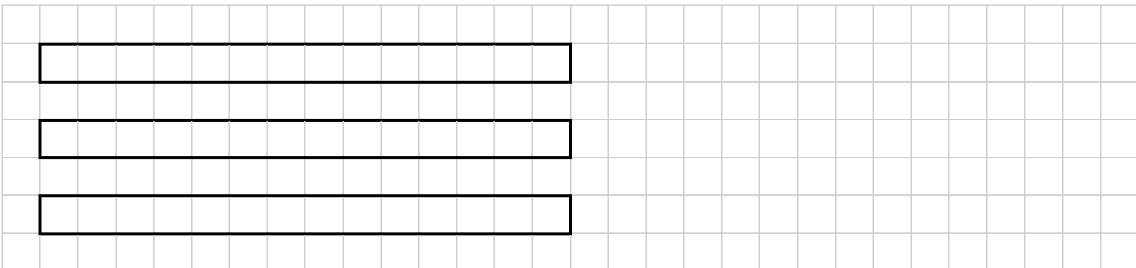
So, it makes sense that $\frac{2}{5} \div \frac{4}{5} = \frac{1}{2}$.

(How many $\frac{4}{5}$s would I need to add up to get $\frac{2}{5}$? 🤔)

Question 1

Use the three identical wholes (rectangles) to show that

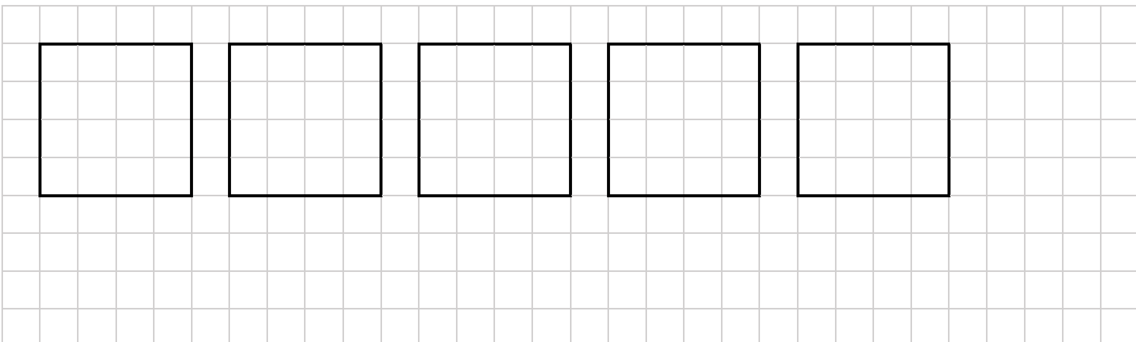
$$3 \div \frac{1}{7} = 21$$



Question 2

Use the 5 identical wholes (squares) to show that

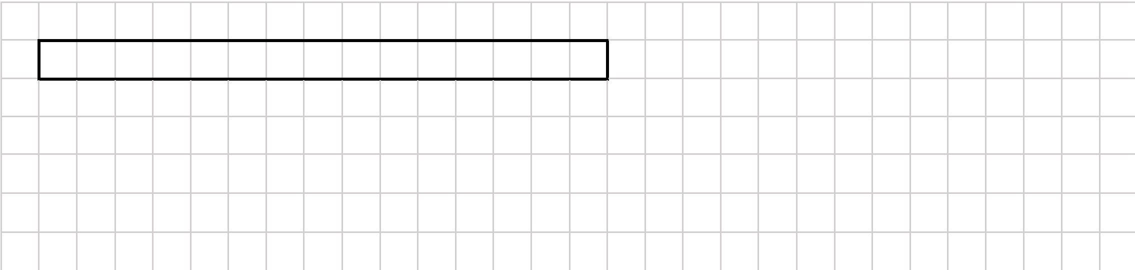
$$5 \div \frac{1}{8} = 40$$



Question 3

Use the rectangle to show that

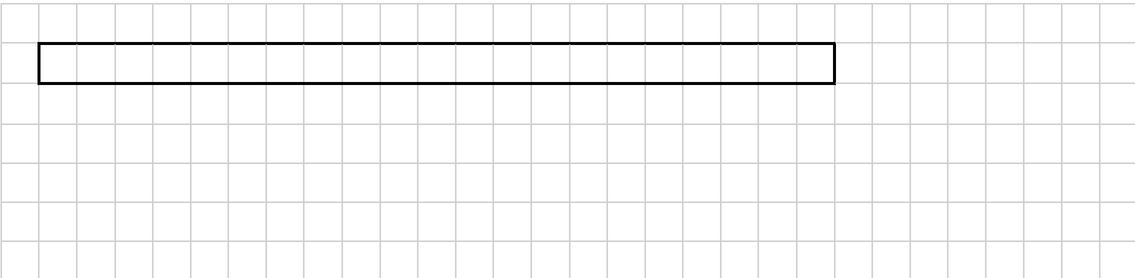
$$\frac{8}{15} \div \frac{2}{15} = 4$$



Question 4

a) Use the rectangle to show that

$$\frac{6}{7} \div \frac{3}{7} = 2$$



b) How many $\frac{3}{7}$s added together will make $\frac{6}{7}$?

Question 5

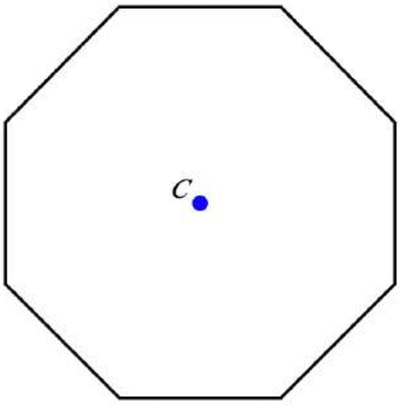
a) Draw a diagram to show that

$$\frac{5}{8} \div \frac{1}{4} = 2\frac{1}{2}$$

b) How many $\frac{1}{4}$s add up to $\frac{5}{8}$?

Question 6

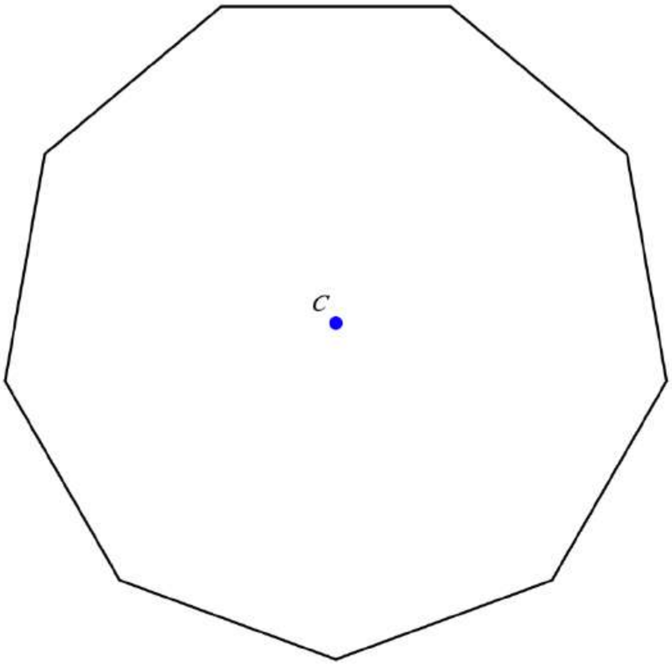
a) Use the regular octagon to show that it makes sense that $\frac{3}{8} \div \frac{5}{8} = \frac{3}{5}$



b) How many $\frac{5}{8}$s added together make $\frac{3}{8}$?

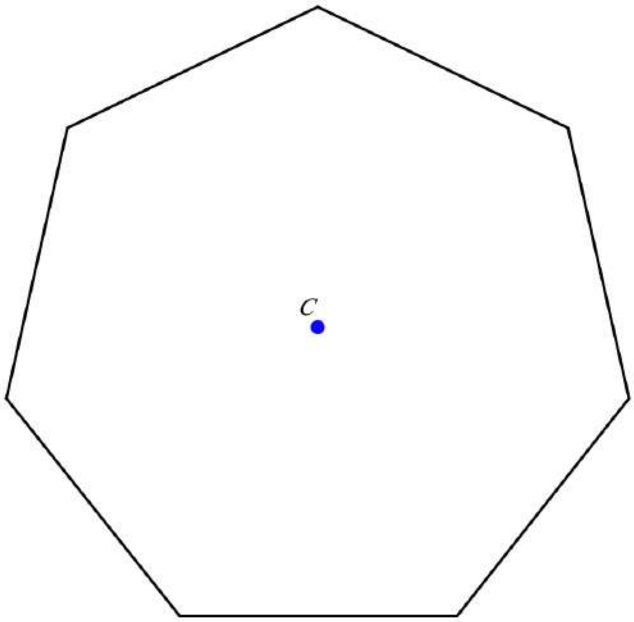
Question 7

Use the regular non-a-gon (9 sides) to show that it makes sense that $\frac{2}{9} \div \frac{8}{9} = \frac{1}{4}$



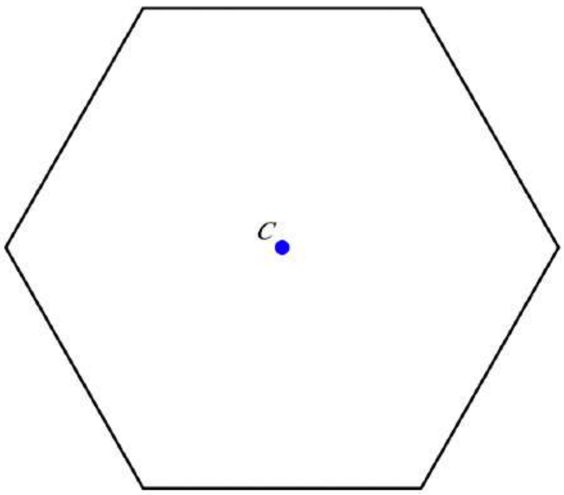
Question 8

Use the regular heptagon (7 sides) to show that it makes sense that $\frac{1}{7} \div \frac{3}{14} = \frac{2}{3}$



Question 9

Use the regular hexagon to show that it makes sense that $\frac{1}{6} \div \frac{7}{18} = \frac{3}{7}$



Question 10

Use two *different* ways, a diagrammatic way and a number-y way, to figure out the result of

$$\frac{3}{4} \div \frac{1}{8}$$

Question 11

Use two *different* ways, a diagrammatic way and a number-y way, to figure out the result of

$$\frac{1}{8} \div \frac{3}{4}$$

22.6 Division involving fractions – the algorithm

In the previous section we determined it made sense that:

$$\frac{3}{5} \div \frac{1}{5} = 3 \qquad \text{and} \qquad \frac{5}{8} \div \frac{1}{4} = \frac{5}{2} \qquad \text{and} \qquad \frac{2}{5} \div \frac{4}{5} = \frac{1}{2}$$

If you were looking for shortcuts for how to calculate the result when dividing one fraction by another, you might have noticed that, *in some cases*, it seems like you can simply divide the numerators and the denominators, and you have the result.

There are various shortcuts that look promising.
Some work in all cases; but, become messy in practice.

There is one shortcut, called the *fraction division algorithm*, that works in all cases and is not too messy! ☺

To explain, we will start with $8 \div 2 = 4$
Note that if we *multiply the dividend and the divisor by the same number*, the result is the same.
E.g.

$$(8 \times 3) \div (2 \times 3)$$
$$= 24 \div 6$$
$$= 4$$

This is because we have multiplied both the *amount we are dividing* and the *amount we are dividing it by,* by the same number (factor) and that factor *divides out*.
This will work no matter the number by which the dividend and the divisor is multiplied.
It is the same idea that underpins equivalent (equal) fractions. ☺

This leads to a lovely idea.

When dividing a fraction by a fraction we could first multiply both the dividend and the divisor by the **reciprocal** of the divisor.
This would result in us dividing by 1, which makes things much easier. ☺
e.g.

$$\frac{3}{5} \div \frac{1}{5}$$
$$= \left(\frac{3}{5} \times \frac{\mathbf{5}}{\mathbf{1}}\right) \div \left(\frac{1}{5} \times \frac{\mathbf{5}}{\mathbf{1}}\right)$$
$$= \left(\frac{3}{5} \times \frac{5}{1}\right) \div 1$$
$$= \frac{3}{5} \times \frac{5}{1}$$

The effect is that our division task is transformed into a multiplication task, *which we already know how to do*!

So, when dividing one fraction by another, we can:

- replace the divisor (the 2nd fraction) with its reciprocal,
- change the operation from division to multiplication,
- calculate the product.

For example.

$$\frac{3}{5} \div \frac{1}{5}$$

$$= \frac{3}{5} \times \frac{5}{1}$$

$$= \frac{15}{5}$$

$$= 3$$

Some people refer to this process as “inverting and multiplying”.

Example 1.

Calculate $\frac{2}{5} \div \frac{4}{5}$

Give the result in simplest form.

$$\frac{2}{5} \div \frac{4}{5}$$

$$= \frac{2}{5} \times \frac{5}{4}$$

$$= \frac{2 \times 5}{5 \times 4}$$

$= \frac{10}{20}$ (Which is NOT in simplest form.)

$= \frac{1}{2}$ (Which is in simplest form.)

Example 2.

Calculate $\frac{2}{5} \div \frac{4}{5}$

Give the result in simplest form.

$$\frac{2}{5} \div \frac{4}{5}$$

$$= \frac{5}{8} \times \frac{4}{1}$$

$$= \frac{5 \times 4}{8 \times 1}$$

$= \frac{20}{8}$ (Which is NOT in simplest form.)

$= \frac{5}{2}$ (Which is in simplest form.)

Example 3.

Calculate $\frac{3}{4} \div \frac{1}{2}$

Give the result in simplest form.

0

$$\frac{3}{4} \div \frac{1}{2}$$

$$= \frac{3}{4} \times \frac{2}{1} \qquad \frac{\overset{1}{\cancel{2}}}{\cancel{2} \times 2}$$

$$= \frac{3 \times 1}{2 \times 1}$$

$$= \frac{3}{2}$$

(Which is in simplest form.)

Example 4.

Calculate $\frac{6}{7} \div \frac{8}{9}$

Give the result in simplest form.

0

$$\frac{6}{7} \div \frac{8}{9}$$

$$= \frac{6}{7} \times \frac{9}{8} \qquad \frac{\overset{1}{\cancel{2}} \times 3}{\cancel{2} \times 2 \times 2}$$

$$= \frac{3 \times 9}{7 \times 4}$$

$$= \frac{27}{28}$$

(Which is in simplest form.)

Example 5.

Calculate $\frac{3}{8} \div 2\frac{1}{12}$

Give the result in simplest form.

0

$$\frac{3}{8} \div 2\frac{1}{12}$$

$$= \frac{3}{8} \div \frac{25}{12}$$

$$= \frac{3}{8} \times \frac{12}{25} \qquad \frac{\overset{1}{\cancel{2 \times 2}} \times 3}{\cancel{2 \times 2} \times 2}$$

$$= \frac{3 \times 3}{2 \times 25}$$

$$= \frac{9}{50}$$

(Which is in simplest form.)

Example 6.

Calculate $\frac{2}{3} \div \frac{8}{9}$

Give the result in simplest form.

0

$$\frac{2}{3} \div \frac{8}{9}$$

$$= \frac{2}{3} \times \frac{9}{8} \qquad \frac{\overset{1}{\cancel{3}} \times 3}{\cancel{3}} \qquad \frac{\overset{1}{\cancel{2}}}{\cancel{2} \times 2 \times 2}$$

$$= \frac{1 \times 3}{1 \times 4}$$

$$= \frac{3}{4}$$

(Which is in simplest form.)

Question 1

Without calculating the result, state whether the result of each is less than 1 or greater than 1.

a) $\frac{1}{4} \div \frac{3}{5}$

b) $\frac{3}{5} \div \frac{1}{8}$

c) $\frac{3}{10} \div \frac{1}{2}$

d) $\frac{1}{8} \div \frac{7}{9}$

e) $1\frac{5}{7} \div \frac{5}{11}$

f) $\frac{6}{7} \div \frac{8}{9}$

Question 2

Calculate each of the following.

a) $\frac{1}{4} \div \frac{3}{5}$

b) $\frac{3}{5} \div \frac{1}{8}$

c) $\frac{3}{10} \div \frac{1}{2}$

d) $\frac{1}{8} \div \frac{7}{9}$

e) $3\frac{1}{2} \div \frac{5}{11}$

f) $\frac{6}{7} \div \frac{8}{9}$

Question 3

Calculate each of the following.

a) $\frac{1}{4} \div \frac{3}{8}$

b) $\frac{3}{5} \div \frac{1}{15}$

c) $\frac{3}{10} \div \frac{9}{11}$

d) $\frac{1}{8} \div \frac{7}{40}$

e) $\frac{5}{2} \div 3\frac{1}{3}$

f) $\frac{6}{7} \div \frac{11}{35}$

Question 4

Calculate each of the following.

a) $\frac{5}{6} \div \frac{3}{8}$

b) $\frac{3}{10} \div \frac{5}{6}$

c) $\frac{4}{5} \div \frac{6}{7}$

d) $\frac{6}{7} \div \frac{8}{9}$

e) $1\frac{3}{7} \div \frac{6}{11}$

f) $\frac{2}{15} \div \frac{3}{10}$

Question 5

Calculate each of the following.

a) $3 \div \frac{3}{5}$

b) $5 \div \frac{5}{7}$

c) $12 \div \frac{2}{9}$

d) $\frac{5}{6} \div 5$

e) $\frac{9}{10} \div 3$

f) $\frac{7}{8} \div 5$

Question 6

Calculate each of the following.

a) $\frac{2}{9} \div \frac{4}{27}$

b) $\frac{3}{14} \div \frac{6}{7}$

c) $\frac{8}{15} \div \frac{2}{3}$

d) $\frac{6}{15} \div \frac{4}{9}$

e) $\frac{6}{10} \div \frac{8}{25}$

f) $\frac{24}{35} \div \frac{18}{49}$

22.7 The fraction division algorithm – explained *differently*

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

Recall that the product of a fraction and its reciprocal (aka multiplicative inverse) is 1. That is,

$$\frac{c}{d} \times \frac{d}{c} = 1$$

We can reason as follows:

$$\frac{a}{b} \div \frac{c}{d}$$

$$= \frac{\frac{a}{b}}{\frac{c}{d}}$$

$$= \frac{\frac{a}{b}}{\frac{c}{d}} \times \frac{\frac{d}{c}}{\frac{d}{c}}$$

$$= \frac{\frac{a}{b} \times \frac{d}{c}}{\frac{c}{d} \times \frac{d}{c}}$$

$$= \frac{\frac{a}{b} \times \frac{d}{c}}{1}$$

$$= \frac{a}{b} \times \frac{d}{c}$$

And so,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

Thus, we can 'invert and multiply' happily.

Some people have been heard to say,

"Ours is not to reason why, just invert and multiply!"

But now we can change the rhyme to,

"Ours IS to reason why, and then forever after, invert and multiply!" ☺

Question 1

Do your best to explain, in your own words, what is happening in each step of the reasoning presented on the previous page and why each step is valid.

22.8 Multiplying and dividing mixed numbers

Numerous methods can be used to multiply/divide mixed numbers. However, the method that will work in all cases is to convert the mixed numbers to improper fractions and proceed using your knowledge of multiplying/dividing fractions. Then, if required, convert the fractional result to a mixed number.

Example 1.

Calculate $2\frac{1}{10} \times 1\frac{2}{3}$

Give the result in simplest mixed number form.

0

$$2\frac{1}{10} \times 1\frac{2}{3}$$

$$= \frac{21}{10} \times \frac{5}{3}$$

$$= \frac{7}{2} \times \frac{1}{1}$$

$$= \frac{7}{2}$$

$=3\frac{1}{2}$ (Which is in simplest form.)

Example 2.

Calculate $1\frac{1}{5} \div 2\frac{2}{3}$

Give the result as a fraction in simplest form.

0

$$1\frac{1}{5} \div 2\frac{2}{3}$$

$$= \frac{6}{5} \div \frac{8}{3}$$

$$= \frac{6}{5} \times \frac{3}{8}$$

$$= \frac{3}{5} \times \frac{3}{4}$$

$= \frac{9}{20}$ (Which is in simplest form.)

Example 3.

Calculate $3\frac{6}{7} \div 3$

Give the result in simplest mixed number form.

0

$$3\frac{6}{7} \div 3$$

$$= \frac{27}{7} \div \frac{3}{1}$$

$$= \frac{27}{7} \times \frac{1}{3}$$

$$= \frac{9}{7} \times \frac{1}{1}$$

$$= \frac{9}{7}$$

$= 1\frac{2}{7}$ (Which is in simplest form.)

Example 4.

Calculate $6\frac{1}{4} \times 4$

Give the result in simplest form.

0

$$6\frac{1}{4} \times 4$$

$$= \frac{25}{4} \times \frac{4}{1}$$

$$= \frac{25}{1} \times \frac{1}{1}$$

$$= \frac{25}{1}$$

$= 25$ (Which is a whole number.)

You might be able to notice a *nicer* way to approach the last two examples. ☺

Question 1

Calculate each of the following. If the result is less than 1, write it as a fraction in simplest form. Otherwise, write it as an improper fraction and a mixed number in simplest form.

a) $1\frac{1}{10} \times 2\frac{1}{3}$

b) $3\frac{2}{5} \div 1\frac{1}{3}$

c) $2\frac{1}{2} \times 2\frac{1}{5}$

d) $1\frac{2}{3} \div 2\frac{1}{5}$

e) $2\frac{1}{5} \div 1\frac{2}{3}$

f) $6\frac{3}{10} \times \frac{2}{5}$

g) $4\frac{4}{5} \div 5$

h) $5 \times 3\frac{2}{9}$

Question 2

Calculate each of the following. If the result is less than 1, write it as a fraction in simplest form. Otherwise, write it as a whole number, or an improper fraction and a mixed number in simplest form, as appropriate.

a) $4\frac{4}{5} \div 1\frac{1}{5}$ b) $3\frac{2}{5} \times 6$ c) $1\frac{2}{7} \div 3\frac{6}{7}$ d) $5 \div 1\frac{1}{5}$

Question 3

Without calculating, state whether:

a) $3\frac{1}{30} \div 3\frac{1}{25}$ is less than or greater than 1 b) $3\frac{3}{8} \times \frac{4}{5}$ is less than or greater than $3\frac{3}{8}$

Question 4

Without calculating exactly, what integer do you think $2\frac{1}{2} \times 2\frac{9}{10}$ is closest too.

22.9 Miscellaneous questions

Question 1

Four-fifths of the marshmallows in a jar of 95 marshmallows are pink. How many pink marshmallows are in the jar?

Question 2

A land developer owns 60 hectares of land and want to split it up into housing blocks $\frac{4}{5}$ of one hectare in size and then sell them.
How many housing blocks will the land developer have for sale?

Question 3

Amanda arrives home to find $\frac{3}{4}$ of her birthday cake remaining in the fridge. She eats $\frac{1}{3}$ of what remains.

a) What fraction of the whole cake did she eat?
b) What fraction of the cake remains?

Question 4

Jenna arrives home to find $\frac{5}{7}$ of her birthday cake remaining in the fridge. She eats $\frac{3}{10}$ of what remains.

a) What fraction of the whole cake did she eat?
b) What fraction of the cake remains now?

Question 5

Roger travels $\frac{3}{5}$ of one kilometre to reach home from school. He ran $\frac{3}{4}$ of the journey.
How many metres did he run?

Question 6

I have a spool of rope that is 36 metres long. I need to create pieces of rope that are $\frac{3}{4}$ of one metre long. How many pieces can I make from the spool?

Question 7

A muffin recipe requires, among other things, $\frac{1}{3}$ of a cup of plain flour, $1\frac{1}{4}$ teaspoons of cinnamon, $\frac{3}{4}$ of a cup of sugar, $2\frac{1}{2}$ teaspoons of baking powder and 55 grams of butter to make 12 muffins. Tayla needs to make 240 muffins.
How much of each of these ingredients will she need?

Question 8

Dean the dog washing guru takes $\frac{3}{4}$ of one hour to wash and pamper a medium sized dog.
How many medium sized dogs can Dean wash in 8 hours?

Question 9

Figure out the numbers that go in each of the blank squares of this multiplication table.
If the number is less than 1, write it as a fraction in simplest form. Otherwise, write it as an improper fraction and mixed number in simplest form.

$\times$		$\frac{1}{6}$	$\frac{3}{4}$	
$\frac{1}{2}$				$\frac{5}{8}$
$\frac{2}{7}$		$\frac{2}{42}=\frac{1}{21}$		
			$\frac{1}{2}$	
$\frac{3}{2}$	$\frac{3}{14}$			

Question 10

Calculate

$$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{5} \times \frac{5}{7} \times \frac{7}{11} \times \frac{11}{13} \times \frac{13}{17} \times \frac{17}{19} \times \frac{19}{23} \times \frac{23}{29} \times \frac{29}{31} \times \frac{31}{37} \times \frac{37}{41} \times \frac{41}{43}$$

Question 11

If *a*, *b*, *c*, and *d* are different digits, find *all the possible ways* to make the following statement true.

$$\frac{a}{b} \times \frac{c}{d} = \frac{5}{6}$$

Question 12

Look at this *never-ending sequence of multiplications* (aka an infinite product):

$$\frac{4}{3} \times \frac{16}{15} \times \frac{36}{35} \times \frac{64}{63} \times \ldots$$

Take your time to calculate, carefully, 20 steps of this infinite product.
Write the result of each step in decimal form (and fractional form if your calculator can).

Do you notice anything interesting about the results?

22.10 Mischievous fractions - revisited

Do you remember those mischievous fractions from an earlier section?

Fractions like, $\frac{4\frac{1}{3}}{6}$ and $\frac{\frac{1}{2}}{\frac{1}{3}}$.

Now that you have developed strong *fraction calculating* skills, you should be able to transform such mischievous monsters into vulgar fractions using a *calculation approach*. ☺

Question 1

Use your *calculation prowess* to transform each of these mischievous fractions to a vulgar fraction of equal value and that is in simplest form.
Before you start, think about whether the result should be less than or greater than 1.

a) $\frac{3\frac{1}{3}}{5}$ b) $\frac{7}{4\frac{1}{5}}$ c) $\frac{\frac{3}{4}}{\frac{2}{3}}$ d) $\frac{\frac{2}{7}}{\frac{4}{21}}$ e) $\frac{2}{3\frac{2}{5}}$ f) $\frac{1}{2-\frac{1}{2-\frac{1}{2-\frac{1}{3}}}}$