

Worked solutions to the

2023 SACE Mathematical Methods Final Examination

Using the fx-CG50AU

These solutions present approaches to the mathematics in this assessment task. These solutions are not exhaustive, and other solutions methods are possible.

These solutions have been prepared by an experienced teacher of SACE mathematics, in-line with published information about SACE assessment practices, but in no way constitute an official set of solutions.



Mathematical Methods

2023

Question booklet 1

- Questions 1 to 6 (52 marks)
- Answer **all** questions
- Write your answers in this question booklet
- You may write on page 10 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

Examination information

Materials

- Question booklet 1
- Question booklet 2
- Formula sheet
- SACE registration number label

Instructions

- Show appropriate working and steps of logic in the question booklets
- State all answers correct to three significant figures, unless otherwise instructed
- Use black or blue pen
- You may use a sharp dark pencil for diagrams and graphical representations

Total time: 130 minutes

Total marks: 100

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Attach your SACE registration number label here

Graphics calculator

1. Brand _____

Model _____

2. Brand _____

Model _____



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Question 1 (7 marks)

Find $\frac{dy}{dx}$ for each of the following functions. There is no need to simplify your answers.

(a) $y = \ln(5x^2 - 2x)$

$\frac{dy}{dx} = \frac{10x-2}{5x^2-2x}$	• use $\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$ (hopefully on your notes sheet) • don't make up log laws - there are none to be used here!
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(2 marks)

(b) $y = \frac{4}{x} + \sqrt{7x+1}$

$y = 4x^{-1} + (7x+1)^{1/2}$ $\frac{dy}{dx} = 4 \times -1 \times x^{-2} + \frac{1}{2}(7x+1)^{-1/2} \times 7$	• writing in "index form" prior to differentiating can be helpful • don't "break the chain" (rule) (by forgetting the 'x7')
--	--

(2 marks)

(c) $y = \frac{\sin(9x-3)}{8e^x}$

$\frac{dy}{dx} = \frac{\cos(9x-3) \times 9 \times 8e^x - \sin(9x-3) \times 8e^x}{(8e^x)^2}$	• using the Quotient Rule • converting to $y = \sin(9x+3) \times (8e^x)^{-1}$ and using the Product Rule
---	--

(3 marks)

Question 2 (8 marks)

Let the random variable T represent the time, measured in minutes, for a randomly chosen customer to receive their order after placing it at a 24-hour fast-food restaurant. T can be modelled using the probability density function

$$f(t) = 0.3e^{-0.3t}, \text{ where } t \geq 0.$$

A graph of $y = f(t)$ is shown in Figure 1.

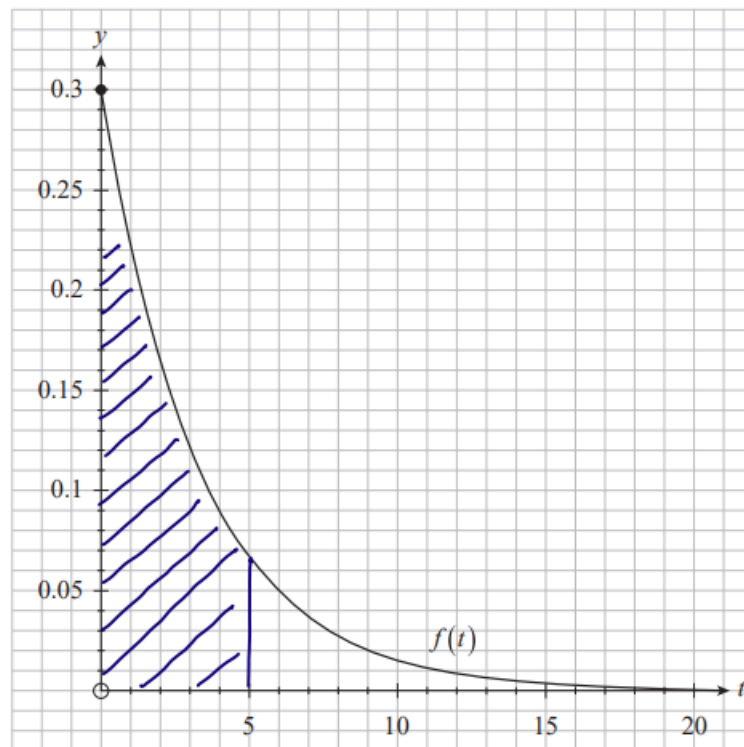


Figure 1

(a) (i) Calculate $\int_2^5 f(t) dt$.

0.326	In Run-Matrix Press OPTN F4 F4
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Math	Rad	Norm2	d/c	Real
$\int_2^5 0.3e^{-0.3x} dx$				
0.3256814759				
Solve d/dx d ² /dt ² 2 f dx SolveN >				

(1 mark)

(ii) Interpret your answer to part (a)(i) in the context of receiving an order at the fast-food restaurant.

The probability of receiving an order within 2 to 5 minutes of ordering.	key ideas: • probability • context i.e. "waiting" or "receiving order" • 2 to 5 mins
--	---

(1 mark)

- (b) (i) Determine the probability that a randomly chosen customer receives their order in less than 5 minutes, according to the model.

$$\int_0^5 F(t) dt = 0.777$$

- Pressing EXE duplicates the previous entry
- Use arrow keys to edit the lower terminal ("2" → "0")

- (ii) On Figure 1, represent the probability calculated in part (b)(i).

Shade region on fig. 1

This is an easy mark to miss!

(1 mark)

- (c) Determine $\int 0.3e^{-0.3t} dt$.

$$= \frac{0.3e^{-0.3t}}{-0.3} + c$$

$$= -e^{-0.3t}$$

- Don't forget the '+c'
 - The simplification shown was not essential, but helps for part (d)
- (1 mark)

- (d) The probability that a randomly chosen customer receives their order in less than r minutes is 0.8, according to the model.

Using your answer to part (c), determine the exact value of r .

$$\int_0^r 0.3e^{-0.3t} dt = 0.8 \quad (1)$$

$$\therefore \left[-e^{-0.3t} \right]_0^r = 0.8 \quad (2)$$

$$-e^{-0.3r} - (-e^{-0.3 \times 0}) = 0.8$$

$$-e^{-0.3r} + 1 = 0.8$$

$$e^{-0.3r} = 0.2$$

$$-0.3r = \ln 0.2$$

$$r = \frac{\ln 0.2}{-0.3}$$

$$\left(= -\frac{10}{3} \ln \frac{1}{5} = \frac{10}{3} \ln 5 \right) \quad (5)$$

(1) convert the words into an equation

(2) use part c

(3) clean-up to form

$$e^A = B$$

(4) solve for r using 'ln'

(5) simplification completely optional

• SolveN on CASIO can solve for decimal solution so will not earn marks (but can be used to check sol)

(3 marks)

Question 3

(7 marks)

Consider the function $f(x) = kx^2 + 1$, where k is a non-zero constant.

(a) Using first principles, show that $f'(x) = 2kx$.

$f'(x) = \lim_{h \rightarrow 0} \frac{k(x+h)^2 + 1 - (kx^2 + 1)}{h}$	• Show use of First principles Formula
$= \lim_{h \rightarrow 0} \frac{\cancel{kx^2} + 2khx + kh^2 + \cancel{1} - \cancel{kx^2} - \cancel{1}}{h}$	• Show binomial expansion and collecting like terms clearly
$= \lim_{h \rightarrow 0} \frac{\cancel{h}(2kx + kh)}{\cancel{h}}$	• Show the cancelling of 'h's' clearly as it is a key step.
$= 2kx$	• use correct limit notation ' $= \lim_{h \rightarrow 0}$ ' up until you let $h=0$

(3 marks)

Figure 2 shows the graph of $y = f(x)$ for a certain value of k . The tangent to the graph at $x = 2$ is also shown. This tangent has a y -intercept at A .

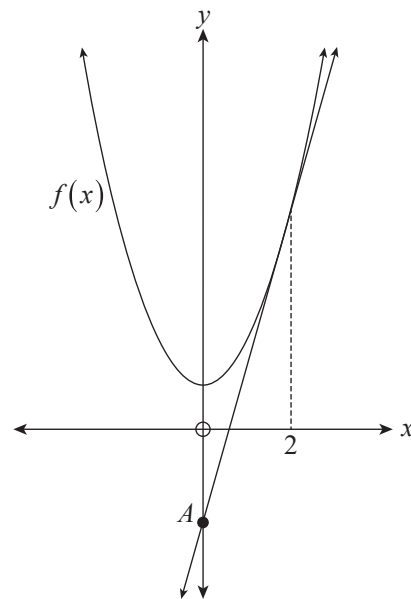


Figure 2

(b) Show that the tangent to the graph of $y = f(x)$ at $x = 2$ has the equation $y = 4kx - 4k + 1$.

	$f(2) = 4k + 1$	• good to make $f(2)$ & $f'(2)$ values clear straight away
	$f'(2) = 4k$	
(*)	$y = 4kx + c$	$4kx - y = 4k(2) - (4k + 1)$
	$\therefore 4k + 1 = 4k \times 2 + c$	$\therefore 4kx - y = 4k - 1$ (*)
	$c = -4k + 1$	$y = 4kx - 4k + 1$
	$\therefore y = 4kx - 4k + 1$	$y = 4k(x - 2) + 4k + 1$
		$= 4kx - 8k + 4k + 1$ (*)
		$= 4kx - 4k + 1$

(3 marks)

• either method fine (others possible)
• As it is a "show" qv, a 'working line' or two need to be shown i.e. (*)

(c) If the y -coordinate of A is -11 , determine the value of k .

y -intercept implies	(1) watch your negative signs
$-4k + 1 = -11$	
$-4k = -12 \dots (1)$	
$k = 3$	

(1 mark)

Question 4 (10 marks)

Astro Cats cards are a type of trading card. Among ordinary Astro Cats cards, gold cards can be found by chance. The probability that a single randomly selected card is a gold card is 0.035. The trading cards are sold in packs of 10 randomly selected cards.



Source: created with assistance from DALL-E 2

- (a) Assume that the number of gold cards found in packs of cards can be modelled using a binomial distribution.

- (i) For **one randomly selected pack** of cards, calculate the:

- (1) expected number of gold cards.

$$X \sim \text{Bin}(10, 0.035)$$

$$E(X) = 10 \times 0.035$$

$$= 0.35$$

(1) Defining your distribution is not essential in an exam

(2) Show the formula used is also not essential for the 1 mark.

(1 mark)

- (2) probability of finding **exactly one** gold card.

$$Pr(X=1)$$

$$= 0.254$$

Binomial P.D
Data : Variable
x : 1
Numtrial : 10
p : 0.035
Save Res: None
Execute
CALC

Binomial P.D
p=0.25398839

(1 mark)

- (3) probability of finding **more than two** gold cards.

$\Rightarrow 3$ or more

$$Pr(X \geq 3)$$

$$= 0.00428$$

Binomial C.D
Data : Variable
Lower : 3
Upper : 10
Numtrial : 10
p : 0.035
Save Res: None
None LIST

Binomial C.D
p=4.2753 $\times 10^{-3}$

(2 marks)

- (ii) A collector states that in **12 randomly selected packs** of cards there is a 100% chance of finding **at least one** gold card.

Calculate the relevant probability to show that the collector's statement is incorrect.

$$Y \sim \text{Bin}(12 \times 10, 0.035)$$

$$Pr(Y \geq 1) = 0.986 < 100\%$$

Binomial C.D
Data : Variable
Lower : 1
Upper : 120
Numtrial : 120
p : 0.035
Save Res: None
None LIST

Binomial C.D
p=0.98609158

(1 mark)

Packs of Astro Cats trading cards may also contain platinum cards. These cards are rare, and the manufacturer does not publicly state how many are made. However, a website claims that the proportion of platinum cards produced is no more than 0.004.

- (b) To assess the accuracy of the website's claim, a collector investigated their 1200 packs of 10 cards and found that these packs contained 51 platinum cards. The collector's 1200 packs of 10 cards can be considered a random sample for the purposes of this question.

- (i) Show that the proportion of platinum cards in this sample is 0.00425.

$\hat{p} = \frac{51}{1200 \times 10}$ $= 0.00425$	• show the fractional form of $\hat{p}$ • don't miss the $\times 10$ due to 10 cards per pack.
---	---

(1 mark)

- (ii) Calculate a 90% confidence interval for the proportion of all

$0.00327 \leq p \leq 0.00523$	• be mindful of correct notation and rounding.
-------------------------------	--

1-Prop ZInterval

C-Level : 0.95

x : 51

n : 1200

Save Res: None

Execute

None

1-Prop ZInterval

Lower=0.03108643

Upper=0.05391356

$\hat{p}$ = 0.0425

n = 1200

- (iii) Hence, at the 90% confidence level, does the confidence interval that you calculated in part (b)(ii) support the website's claim? Justify your answer.

No, as the confidence interval contains values above 0.004	• answer the yes/no question then give a justification • Sometimes 'less is more'. Do <u>not</u> add extra declarations like "so the claim is false" (it is not proved false by the CI, it is just 'not supported')
---	--

(2 marks)

Question 4 (10 marks)

Astro Cats cards are a type of trading card. Among ordinary Astro Cats cards, gold cards can be found by chance. The probability that a single randomly selected card is a gold card is 0.035. The trading cards are sold in packs of 10 randomly selected cards.



Source: created with assistance from DALL-E 2

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$$E(X) = 10 \times 0.035$$

$$= 0.35$$

(1) Defining your distribution is not essential in an exam

(2) Show the formula used is also not essential for the 1 mark.

(1 mark)

- (2) probability of finding **exactly one** gold card.

$$Pr(X=1)$$

$$= 0.254$$

Binomial

Tail : Δ (X=)

x : 1

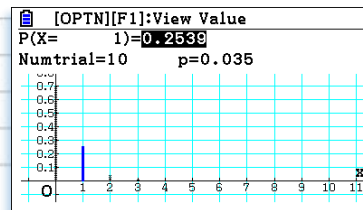
Numtrial : 10

p : 0.035

Execute

EXE

DETAIL



(1 mark)

- (3) probability of finding **more than two** gold cards.

$\Rightarrow$ 3 or more

$$Pr(X \geq 3)$$

$$= 0.00428$$

Binomial

Tail : Δ (X $\geq$)

x : 3

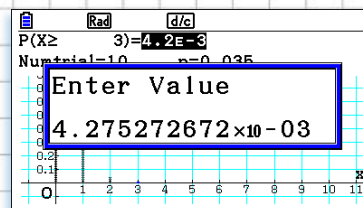
Numtrial : 10

p : 0.035

Execute

EXE

DETAIL



(2 marks)

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Calculate the relevant probability to show that the collector's statement is incorrect.

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Binomial

Tail : Δ (X $\geq$)

x : 1

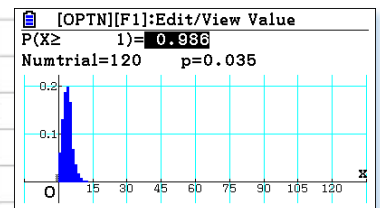
Numtrial : 120

p : 0.035

Execute

EXE

DETAIL



(1 mark)

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(1 mark)

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C-Level : 0.95

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Save Res: None

Execute

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No, as the confidence interval contains values above 0.004	• answer the yes/no question then give a justification • Sometimes 'less is more'. Do <u>not</u> add extra declarations like "so the claim is false" (it is not proved false by the CI, it is just 'not supported')
---	--

(2 marks)

Question 5

(12 marks)

- (a) Consider the function $f(x) = \frac{2}{x-1} + 2x$, where $x \neq 1$. A graph of $y = f(x)$ is shown in Figure 3, with its two stationary points shown at A and B .

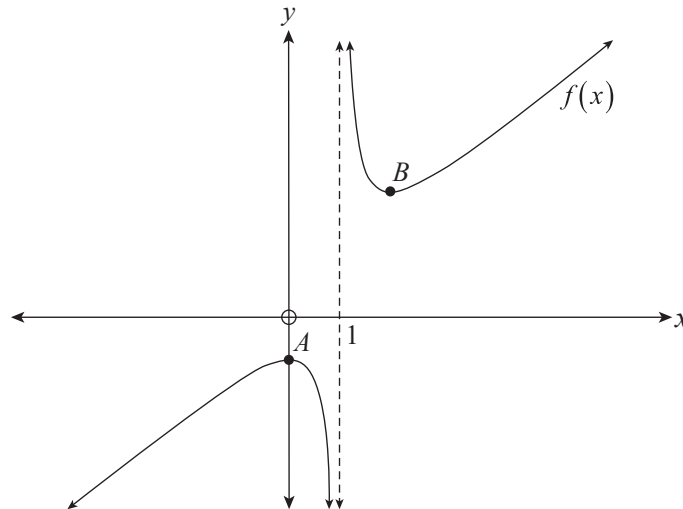


Figure 3

- (i) Show that $f'(x) = \frac{2x^2 - 4x}{(x-1)^2}$.

$f(x) = 2(x-1)^{-1} + 2x$	(1) better to use Chain Rule than to convert to $\frac{2+2x(x-1)^2}{(x-1)^2}$ and use QR.
$\therefore f'(x) = -2(x-1)^{-2} + 2$	(2) Show the making of a common denominator
$= \frac{-2}{(x-1)^2} + 2 \times \frac{(x-1)^2}{(x-1)^2}$	(3) Watch your binomial expansion
$= \frac{-2 + 2x^2 - 4x + 2}{(x-1)^2}$	

(3 marks)

- (ii) Hence, use an algebraic approach to show that $f(x)$ has exactly two stationary points, at $x = 0$ and $x = 2$.

$f'(x) = 0$	(1) make your algebraic approach clear i.e. Factorising or the Quadratic Formula.
$\Rightarrow 2x^2 - 4x = 0$	
$2x(x-2) = 0$ (1)	
$x = 0$ or $x = 2$	

(2 marks)

(b) Consider the function $g(x) = \frac{2}{x-m} + 2x$, where m is a constant and $x \neq m$.

(i) Complete the following table for each given value of m .

m	Function	x -coordinates of the two stationary points of the function $g(x)$
1	$g(x) = \frac{2}{x-1} + 2x$	$x = 0, x = 2$
2	$g(x) = \frac{2}{x-2} + 2x$	$x = 1, x = 3$
3	$g(x) = \frac{2}{x-3} + 2x$	$x = 2, x = 4$
4	$g(x) = \frac{2}{x-4} + 2x$	$x = 3, x = 5$

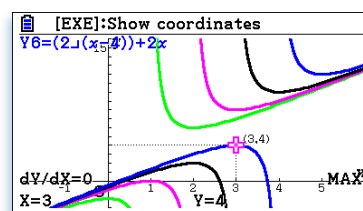
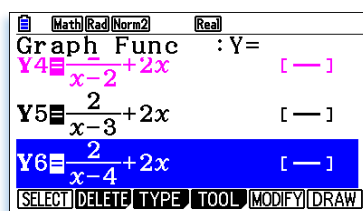
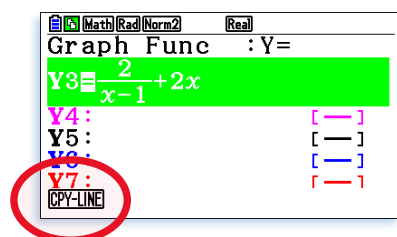
• Best done graphically
• "Clip and Paste" can speed up function entry (SHIFT + [8] or [9])
• adjust view window (2 marks) as required

(ii) Make a conjecture about the x -coordinates of the two stationary points of the function

$$g(x) = \frac{2}{x-m} + 2x.$$

$x = m-1, x = m+1$

(1 mark)



(iii) Prove or disprove your conjecture.

$$g'(x) = \frac{-2}{(x-m)^2} + 2 = 0$$

$$\text{when } -2 + 2(x-m)^2 = 0$$

$$\therefore (x-m)^2 = 1$$

$$x-m = \pm 1 \quad (*)$$

$$x = m \pm 1$$

_____ or _____

$$g'(x) = \frac{-2}{(x-m)^2} + 2 = 0$$

$$\text{when } -2 + 2(x-m)^2 = 0$$

$$2x^2 - 4mx + 2m^2 - 2 = 0 \quad (+)$$

$$\therefore x = \frac{-(-4m) \pm \sqrt{(-4m)^2 - 4(2)(2m^2-2)}}{2(2)}$$

$$= \frac{4m \pm \sqrt{16m^2 - 16m^2 + 16}}{4}$$

$$= \frac{4m \pm 4}{4}$$

$$= m \pm 1$$

• Set $g'(x)=0$ leads to a quadratic equation

• Solving via method 1 is elegant but don't miss the $\pm$ (*)

• Expanding and using the quadratic formula is less elegant but can be effective.

• Dividing through by 2 would have made the quadratic formula work easier (+)

(4 marks)

Question 6 begins on page 14.

Question 6

(8 marks)

Consider the graph of the function $y = f(x)$ shown in Figure 4 where $f(x)$ is defined for all real values of x . This function has exactly:

- two stationary points, located at $B(b, f(b))$ and $D(d, f(d))$
- two inflection points, located at $A(a, f(a))$ and $C(c, f(c))$
- two x -intercepts, located at $B(b, f(b))$ and $E(e, f(e))$.

As x increases for $x > d$, the slope of $f(x)$ approaches k , where $k < 0$. Additionally, $f''(x) = 0$ only at $x = a$ and $x = c$.

important for part (d)

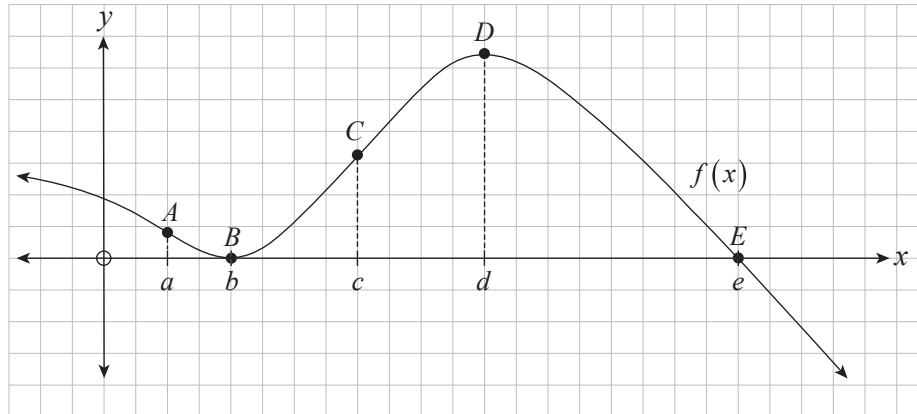
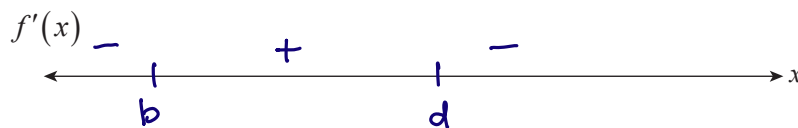


Figure 4

(a) Complete the sign diagram below for $f'(x)$.



Do not include non-critical values i.e. a, c, e on your Sign diagram

(2 marks)

(b) (i) Which *one* statement is true? Tick the appropriate box to indicate your answer.

$f'(0) < f'(a)$ ☐

$f'(0) = f'(a)$ ☐

$f'(0) > f'(a)$ ☒

• As slope at A is steeper but negative

(1 mark)

(ii) Which *one* statement is true? Tick the appropriate box to indicate your answer.

$f(d) < 0$ and $f''(d) < 0$ ☐

$f(d) < 0$ and $f''(d) > 0$ ☐

$f(d) > 0$ and $f''(d) < 0$ ☒

$f(d) > 0$ and $f''(d) > 0$ ☐

• as D is above the x-axis and in a concave (convex down) region of the curve.

(1 mark)

(c) State the interval(s) of x for which $f''(x) \geq 0$.

$a \leq x \leq c$	• use 'inclusive inequalities' because the question does.
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(1 mark)

(d) On Figure 5, sketch a possible graph of $y = f'(x)$. Note that k has been added to the y -axis.

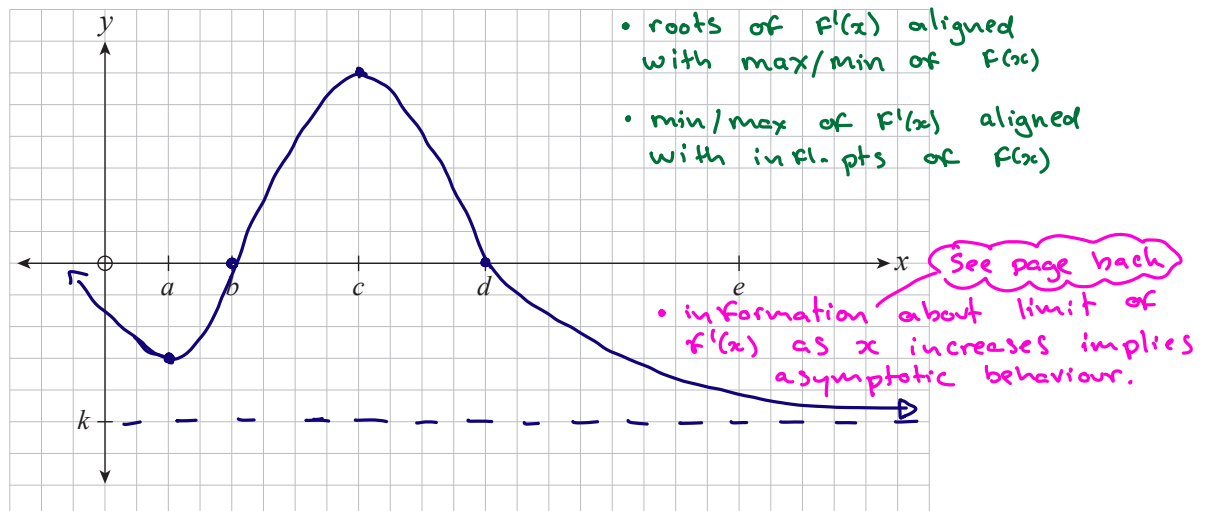


Figure 5

(3 marks)

Mathematical Methods

2023

Question booklet 2

- Questions 7 to 10 (48 marks)
- Answer **all** questions
- Write your answers in this question booklet
- You may write on pages 9 and 12 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

2

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SEQ	FIGURES	CHECK LETTER	BIN

Graphics calculator

1. Brand

Model

2. Brand

Model

Question 7 (14 marks)

The daily milk production of a randomly chosen milking cow varies, and can be modelled by a normal distribution with a mean of $\mu = 21.9$ litres and a standard deviation of $\sigma = 3.26$ litres.



Source: wernerdejten | pixabay.com

- (a) (i) Determine the probability that the daily milk production of a randomly selected cow is less than 23 litres.

$X \sim N(21.9, 3.26^2)$ $\Pr(X < 23)$ $= 0.632$		
--	--	--

(1 mark)

- (ii) Given that 15% of randomly selected cows have a daily milk production of c litres or more, determine the value of c .

$\Pr(X \geq c) = 0.15$ $\therefore c = 25.3$		
--	--	--

(1 mark)

- (b) Let the random variable S_n represent the total daily milk production of a random sample of n cows. Explain why it is appropriate to model S_n using a normal distribution.

As milk production of <u>individual</u> cows is <u>normally distributed</u>, then so is the <u>distribution of sample sums</u> (of any size)	• <u>key idea!</u> the population of individuals is normally distributed.
---	--

(1 mark)

Recall that the daily milk production of a randomly chosen milking cow can be modelled by a normal distribution with a mean of $\mu = 21.9$ litres and a standard deviation of $\sigma = 3.26$ litres.

- (c) Let the random variable S_{90} represent the total daily milk production of a random sample of 90 cows.

- (i) Show that the mean of S_{90} is $\mu_{S_{90}} = 1971$ litres and standard deviation of S_{90} is $\sigma_{S_{90}} = 30.93$ litres (correct to four significant figures).

$\mu_{S_{90}} = 90 \times 21.9$ (*)	$\sigma_{S_{90}} = \sqrt{90} \times 3.26$ (*)	• key is to show the calculations required (*)
$= 1971$	$= 30.927$ (+)	
	$= 30.93$ (4sf)	• nice to show the extra figure (+) where possible. (2 marks)

- (ii) Determine the probability that the total daily milk production of a random sample of 90 cows is:

- (1) between 1900 litres and 1950 litres.

$$Pr(1900 < S_{90} < 1950) = 0.238$$

	Rad(Norm2)	d/c(Real)
Normal C.D		
Data	:Variable	
Lower	:1900	
Upper	:1950	
σ	:30.9270755	
μ	:1971	
Save Res:None		
None	LIST	

Rad	Norm2	d/c	Real
Normal C.D			
p	=0.23771778		
z:Low	=-2.295723		
z:Up	=-0.6790166		

- (2) more than 2000 litres.

$$Pr(S_{90} > 2000) = 0.174$$

	Rad(Norm2)	d/c(Real)
Normal C.D		
Data	:Variable	
Lower	:2000	
Upper	:1×10 ⁹	
σ	:30.9270755	
μ	:1971	
Save Res:None		
None	LIST	

Rad	Norm2	d/c	Real
Normal C.D			
p	=0.17420195		
z:Low	=0.93768969		
z:Up	=3.2334 $\times 10^7$		

A farmer owns 90 cows, which can be considered a random sample for the purposes of this question. The farmer aims to increase their cows' total mean daily milk production by adopting a new feeding plan for the cows.

The farmer randomly selects 20 cows for the new feeding plan. In this random sample of 20 cows, the mean daily milk production after some time on the new feeding plan is 23.5 litres per cow.

- (d) Calculate a 95% confidence interval for the mean daily milk production per cow on the new feeding plan. Assume that the standard deviation is still $\sigma = 3.26$ litres.

$$22.1 \leq \mu \leq 24.9$$

	Rad	Norm2	d/c	Real
1-Sample ZInterval				
Data	:	Variable		
C-Level	:	0.95		
σ	:	3.26		
$\bar{x}$	:	23.5		
n	:	20		
Save Res:None				
None	LIST			

Rad	Norm2	d/c	Real
1-Sample ZInterval			
Lower	=22.0712683		
Upper	=24.9287317		
$\bar{x}$	=23.5		
n	=20		

- (e) Does the confidence interval calculated in part (d) suggest with 95% confidence that the new feeding plan has increased the mean daily milk production per cow? Justify your answer.

Yes, as the 95% confidence interval	• answer the yes/no (does CI suggest increase)
for the new mean lies wholly	• give reason, the spatial/ numerical relationship
above the old mean of 21.9	between the CI and the old mean. (2 marks)

- (f) The farmer would like the 90 cows' mean total daily milk production to be more than 2000 litres.

- (i) Determine the mean daily milk production per cow if 90 cows had a total daily milk production of 2000 litres.

$\frac{2000}{90}$	• turn a total of 2000 into an average of 22.2
= 22.2	

(1 mark)

- (ii) Hence or otherwise, does the confidence interval calculated in part (d) suggest with 95% confidence that the new feeding plan will result in the 90 cows' mean total daily milk production being more than 2000 litres? Justify your answer.

No, as the confidence interval contains values	• answer the yes/no (no, does not suggest a total more than 2000)
less than 22.2, corresponding to less than	• give reason, the CI contains value below the target of 2000+
2000 litres being produced.	(2 marks)



Question 7 (14 marks)

The daily milk production of a randomly chosen milking cow varies, and can be modelled by a normal distribution with a mean of $\mu = 21.9$ litres and a standard deviation of $\sigma = 3.26$ litres.



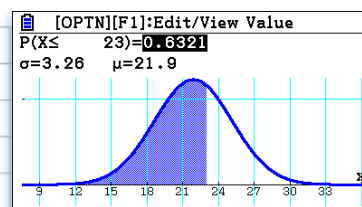
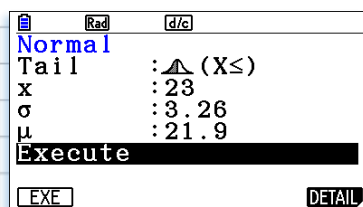
Source: wernerdetjen | pixabay.com

- (a) (i) Determine the probability that the daily milk production of a randomly selected cow is less than 23 litres.

$$X \sim N(21.9, 3.26^2)$$

$$\Pr(X < 23)$$

$$= 0.632$$

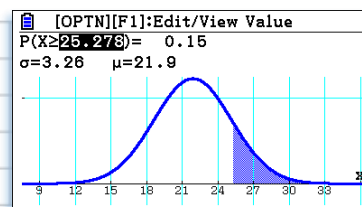
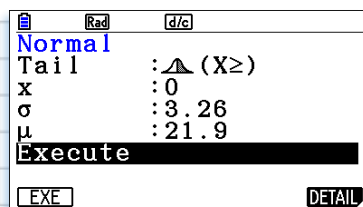


(1 mark)

- (ii) Given that 15% of randomly selected cows have a daily milk production of c litres or more, determine the value of c .

$$\Pr(X \geq c) = 0.15$$

$$\therefore c = 25.3$$



(1 mark)

To perform this 'inverse' calculation, switch the tail as required (use any x-value) then enter the probability in the subsequent screen

- (b) Let the random variable S_n represent the total daily milk production of a random sample of n cows. Explain why it is appropriate to model S_n using a normal distribution.

As milk production of individual cows is normally distributed, then so is the distribution of sample sums (of any size)

• key idea!
the population of individuals is normally distributed.

(1 mark)

Recall that the daily milk production of a randomly chosen milking cow can be modelled by a normal distribution with a mean of $\mu = 21.9$ litres and a standard deviation of $\sigma = 3.26$ litres.

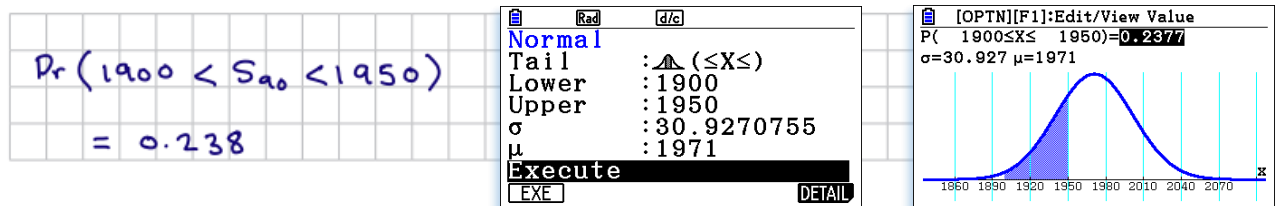
- (c) Let the random variable S_{90} represent the total daily milk production of a random sample of 90 cows.

- (i) Show that the mean of S_{90} is $\mu_{S_{90}} = 1971$ litres and standard deviation of S_{90} is $\sigma_{S_{90}} = 30.93$ litres (correct to four significant figures).

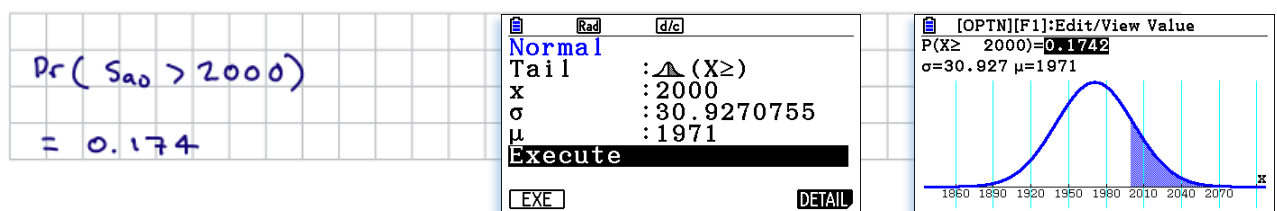
$\mu_{S_{90}} = 90 \times 21.9$ (*)	$\sigma_{S_{90}} = \sqrt{90} \times 3.26$ (*)	• key is to show the calculations required (*) • nice to show the extra figure (+) where possible. (2 marks)
$= 1971$	$= 30.927$ (+)	
	$= 30.93$ (4sf)	

- (ii) Determine the probability that the total daily milk production of a random sample of 90 cows is:

- (1) between 1900 litres and 1950 litres.



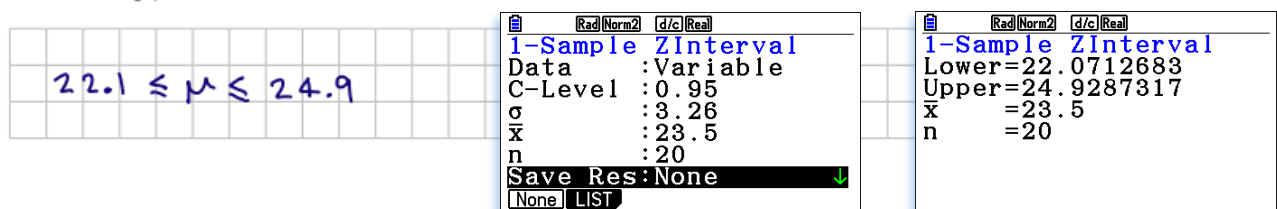
- (2) more than 2000 litres.



A farmer owns 90 cows, which can be considered a random sample for the purposes of this question. The farmer aims to increase their cows' total mean daily milk production by adopting a new feeding plan for the cows.

The farmer randomly selects 20 cows for the new feeding plan. In this random sample of 20 cows, the mean daily milk production after some time on the new feeding plan is 23.5 litres per cow.

- (d) Calculate a 95% confidence interval for the mean daily milk production per cow on the new feeding plan. Assume that the standard deviation is still $\sigma = 3.26$ litres.



- (e) Does the confidence interval calculated in part (d) suggest with 95% confidence that the new feeding plan has increased the mean daily milk production per cow? Justify your answer.

Yes, as the 95% confidence interval	• answer the yes/no (does CI suggest increase)
for the new mean lies wholly	• give reason, the spatial/ numerical relationship
above the old mean of 21.9	between the CI and the old mean. (2 marks)

- (f) The farmer would like the 90 cows' mean total daily milk production to be more than 2000 litres.

- (i) Determine the mean daily milk production per cow if 90 cows had a total daily milk production of 2000 litres.

$\frac{2000}{90}$	• turn a total of 2000 into an average of 22.2
$= 22.2$	

(1 mark)

- (ii) Hence or otherwise, does the confidence interval calculated in part (d) suggest with 95% confidence that the new feeding plan will result in the 90 cows' mean total daily milk production being more than 2000 litres? Justify your answer.

No, as the confidence interval contains values	• answer the yes/no (no, does not suggest a total more than 2000)
less than 22.2, corresponding to less than	• give reason, the CI contains value below the target of 2000+
2000 litres being produced.	(2 marks)

Question 8 (10 marks)

A self-balancing robot uses a device called a PID controller to stop itself from falling over after being placed on its wheels and released. Let $L(t)$ represent the angle of lean of the robot, at t seconds after release, where $L(t)$ is measured in **radians**. Side-on diagrams of this robot in Figure 6 show various angles of lean, where:

- negative values of $L(t)$ mean the robot has an angle of lean to the left
- $L(t) = 0$ means the robot is vertical
- positive values of $L(t)$ mean the robot has an angle of lean to the right.

This image cannot be reproduced here for copyright reasons.

Source: www.superdroidrobots.com

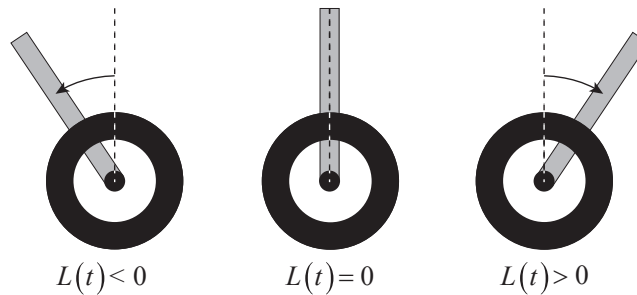


Figure 6

The PID controller's settings can be adjusted to change how long it takes the robot to reach an approximately stable vertical position after initially being placed on its wheels and released.

Engineers conducted an experiment using different settings on the PID controller, starting with Setting A.

- (a) Using Setting A, the robot was initially placed on its wheels and released. The resulting angle of lean was recorded for 6 seconds.

Using this setting, the angle of lean of the robot, $L(t)$, at t seconds after release, can be modelled by the function

$$L(t) = 0.75e^{-0.75t} \cos(4t), \text{ where } 0 \leq t \leq 6.$$

A graph of $y = L(t)$ for Setting A is shown in Figure 7.

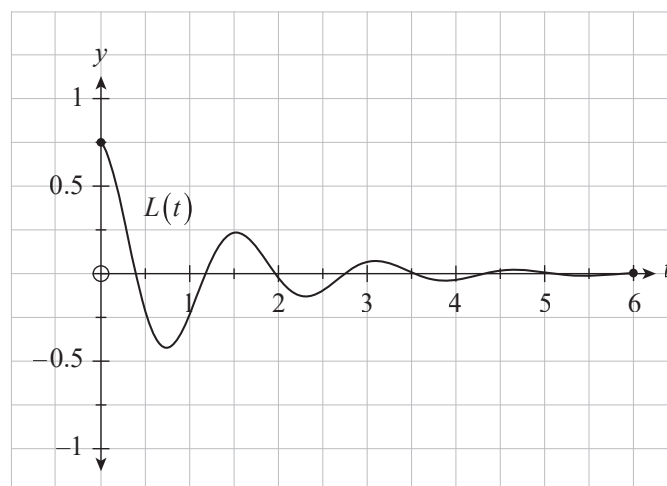
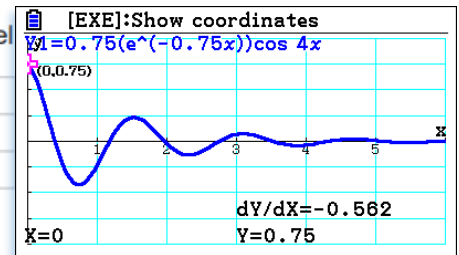


Figure 7

(i) State the initial angle of lean of the robot, according to the model

0.75

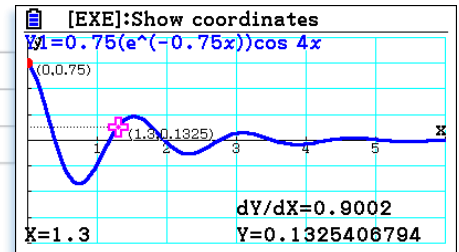
• Use GC cos you'll need it for (ii) and (iv), but otherwise could sub in easily by hand (and check on graph provided)



(ii) Determine the value of $L'(1.3)$.

0.900

• Use Graph in G.C. Trace to $x=1.3$ make sure Derivative is ON



(iii) Interpret your answer to part (a)(ii) in the context of the problem, stating the appropriate units.

1.3 seconds after release, the angle of lean is changing at 0.9 radians per second.

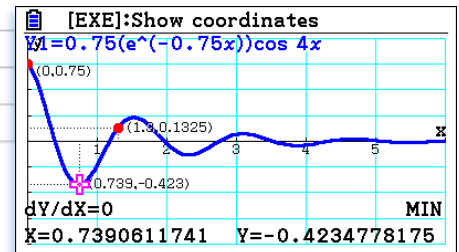
key ideas:
 • time
 • quantity (angle)
 • idea for 'change' or rate of change
 • units

(2 marks)

(iv) Calculate how long after being released the robot is leaning to the left with its maximum angle, according to the model.

$t = 0.739$ secs

• Greatest lean left $\Rightarrow L(t)$ is minimised as negative L values represent leaning left.



Question 8 continues on page 8.

- (b) The engineers were not satisfied with how long the robot took to reach an approximately stable vertical state using Setting A. To improve results, the experiment was repeated using Setting B and the same initial angle of lean.

Using Setting B, the angle of lean of the robot, $L(t)$, at t seconds after release, can now be modelled by the new function

$$L(t) = 0.75e^{-t} \cos t, \text{ where } 0 \leq t \leq 6.$$

A graph of $y = L(t)$ for Setting B is shown in Figure 8.

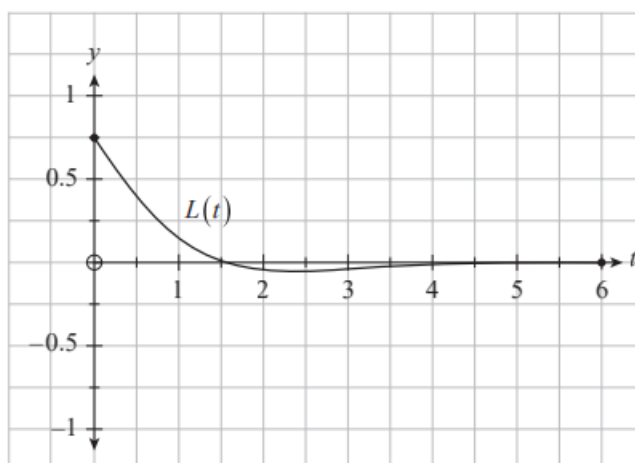


Figure 8

- (i) Determine $L'(t)$ for Setting B.

$$L'(t) = 0.75e^{-t} \times -1 \times \cos t + 0.75e^{-t} \times -\sin t$$

• use product rule

(2 marks)

- (ii) Hence, for Setting B, use an algebraic approach to determine the exact values of t such that $L'(t) = 0$, during the first 6 seconds after release.

⊗

$$\begin{cases} L'(t) = 0 \text{ when } -e^{-t}(\cos t + \sin t) = 0 \\ \therefore \cos t + \sin t = 0 \\ \cos t = -\sin t \\ \tan t = -1 \end{cases}$$

$t = \frac{3\pi}{4}, \frac{7\pi}{4}$

- Algebraic approach required factoring $L'(t)$ and setting trig. eqn equal to zero. (✗)
- Resulting equation could be solved with the help of a unit circle (or solver)

Math Rad Norm2 d/c Real

$(\cos x = -\sin x, x, 0, 6)$

$\left\{ \frac{3\pi}{4}, \frac{7\pi}{4} \right\}$

Solve d/dx d²/dx² f dx SolveN

Math Rad Norm2 d/c Real

SolveN(cos x = -sin x, $\left\{ \frac{3\pi}{4}, \frac{7\pi}{4} \right\}$)

$\tan^{-1}(-1)$

$-\frac{1}{4}\pi$

Solve d/dx d²/dx² f dx SolveN

Math Rad Norm2 d/c Real

Ans+π $\frac{3\pi}{4}$

Ans+π $\frac{7\pi}{4}$

Solve d/dx d²/dx² f dx SolveN

Question 9 (10 marks)

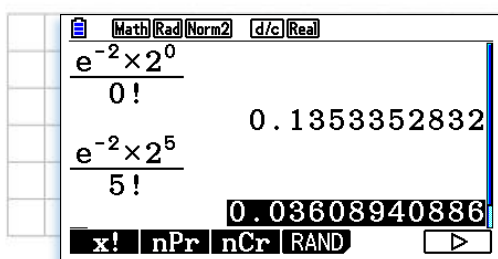
Consider a discrete random variable, X , which has the probability mass function

$$\Pr(X = x) = \frac{e^{-2} 2^x}{x!},$$

where x is a non-negative integer, and $x! = x \times (x-1) \times (x-2) \times \dots \times 3 \times 2 \times 1$ with $0! = 1$.

- (a) Based on the information provided, complete the table below by calculating the missing probabilities. State your answers correct to four decimal places.

$\Pr(X=0)$	$\Pr(X=1)$	$\Pr(X=2)$	$\Pr(X=3)$	$\Pr(X=4)$	$\Pr(X=5)$	$\Pr(X \geq 6)$
0.1353	0.2707	0.2707	0.1804	0.0902	0.0361	0.0166



The factorial symbol (!) is entered by pressing OPTN F6 F3 F1
Pressing EXE again duplicates the entry above which can then be edited as required.

(2 marks)

A moment-generating function can be used to find key properties of a distribution without using the probability mass function directly.

For the probability mass function given above, the moment-generating function is

$$M(t) = e^{2e^t - 2}.$$

- (b) The mean, μ , of the probability mass function given above can be found using the formula

$$\mu = M'(0).$$

Show that $\mu = 2$.

$$\begin{aligned} M'(t) &= e^{2e^t - 2} \times 2e^t \\ \mu &= M'(0) = e^{2e^0 - 2} \times 2e^0 \\ &= e^{2-2} \times 2 \times 1 \quad (*) \\ &= 1 \times 2 \times 1 = 2 \end{aligned}$$

- Use chain rule, be careful of the exponents.
- Sub in $t=0$ (don't solve) $M'(t)=0$
- due to 'show' a 'working line' is recommended i.e. (*)

(2 marks)

(c) (i) Show that $M''(t) = 2e^{2e^t-2}(2e^{2t} + e^t)$.

$$\begin{aligned}
 M''(t) &= e^{2e^t-2} \times 2e^t \times 2e^t + e^{2e^t-2} \times 2e^t \\
 &= e^{2e^t-2} (4e^{2t} + 2e^t) \\
 &= 2e^{2e^t-2} (2e^{2t} + e^t)
 \end{aligned}$$

• product and chain rules needed (with care)

(2 marks)

(ii) The standard deviation, σ , of the probability mass function given above can be found using the formula

$$\sigma = \sqrt{M''(0) - \mu^2}.$$

Show that $\sigma = \sqrt{2}$.

$$\begin{aligned}
 M''(0) &= 2e^{2e^0-2} (2e^{2 \times 0} + e^0) \\
 &= 2 \times 1 \times (2 + 1) \\
 &= 6 \\
 \therefore \sigma &= \sqrt{6 - 2^2} \\
 &= \sqrt{2}
 \end{aligned}$$

• Show the substitution of $t=0$ in $M''(t)$

• Show the sub into σ formula

(both of the above can be done simultaneously)

(2 marks)

(d) Hence, determine the probability that X is within one standard deviation of the mean.

$$\begin{aligned}
 &\approx 0.5\% \quad \Pr(\mu - \sigma < x < \mu + \sigma) \\
 &= \Pr(2 - \sqrt{2} < x < 2 + \sqrt{2}) \quad \leftarrow \approx 3.41 \\
 &= \Pr(x = 1, 2, \text{ or } 3) \quad \leftarrow \text{as } x \text{ is integer} \\
 &= 0.2707 + 0.2707 + 0.1804 \\
 &= 0.7218
 \end{aligned}$$

• Calculating $\mu \pm \sigma$ and realising that this means $x = 1, 2, 3$ due to its integer nature

• the prob. calculation can be done by summing values from the previous table.

(2 marks)

Question 10 (14 marks)

Consider the function $f(x) = \ln(2(x+1)^2)$ for $x > -1$. The graph of $y = f(x)$ is shown in Figure 9.

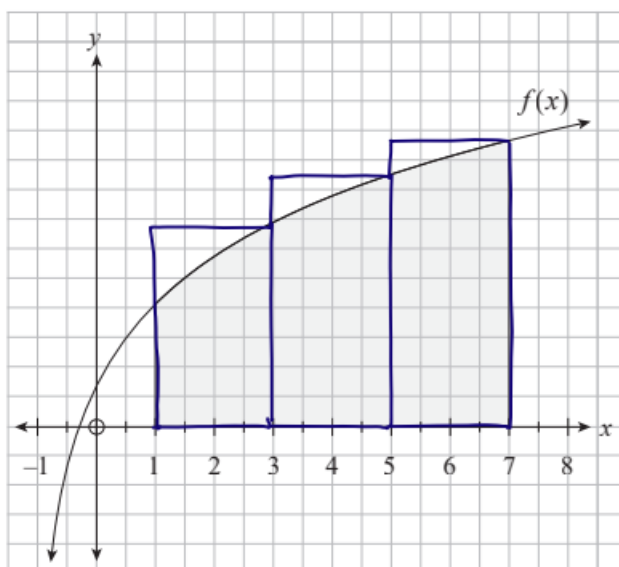


Figure 9

Let A be the area bounded by the graph of $y = f(x)$, the x -axis, and the vertical lines at $x = 1$ and $x = 7$. In Figure 9, A is represented by the shaded region.

(a) Let U_3 represent an upper estimate of A using the areas of three rectangles of equal width.

(i) On Figure 9, draw three rectangles that can be used to calculate U_3 .

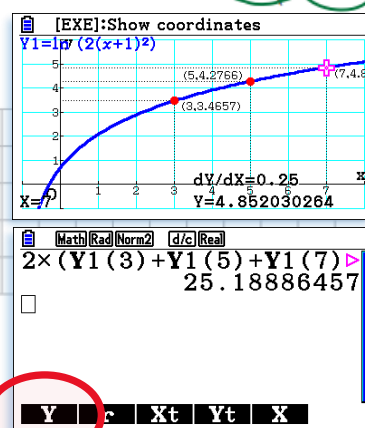
don't miss this easy mark!
(1 mark)

(ii) Hence, calculate the value of U_3 .

$$U_3 = 2 \times 3.466 + 2 \times 4.277 + 2 \times 4.852$$

$$= 25.2$$

watch the width = 2



(iii) Write an integral expression for the exact value of A .

$$\int_1^7 f(x) dx$$

For part (a)(ii), draw the graph to ensure that the function is entered correctly. Trace to $x=3, 5, 7$ to obtain the function values required.

These values can be called up in Run-Matrix as shown above, pressing **VARs F4 F1** to enter Y. (1 mark)

(b) Consider the function $y = x \ln x$, where $x > 0$.

(i) Find $\frac{dy}{dx}$.

$\frac{dy}{dx} = 1 \times \ln x + x \times \frac{1}{x}$	• a familiar way to obtain an 'unfamiliar' integral as needed for part (b)(ii)
$= \ln x + 1$	

(1 mark)

(ii) Using your result from part (b)(i), show that

$$\int \ln(2x^2) dx = x(\ln(2x^2) - 2) + c,$$

where c is a real constant and $x > 0$.

From (b)(i) $\int \ln x + 1 dx = x \ln x + c$ $\therefore \int \ln x dx = x \ln x - \int 1 dx + c$ $= x \ln x - x + c \dots (1)$	Being clear about how part (i) relates to integrals is good practice.
$\int \ln(2x^2) dx$ $= \int \ln 2 + \ln x^2 dx$ $= \int \ln 2 dx + \int 2 \ln x dx$ $= (\ln 2)x + 2(x \ln x - x) + c \quad (\text{using (1)})$ $= x(\ln 2 + 2 \ln x - 2) + c$ $= x(\ln(2x^2) - 2) + c$ $= x(\ln(2x^2) - 2) + c$	Split up the integral using integral rules and log laws Show the sub in of the part (i) result Show the simplification required to get the desired result i.e. factoring out the 'x'

(4 marks)

Consider the functions $f(x) = \ln(2(x+1)^2)$ for $x > -1$ and $g(x) = \ln(2x^2)$ for $x > 0$.
The graphs of $y = f(x)$ and $y = g(x)$ are shown in Figure 10.

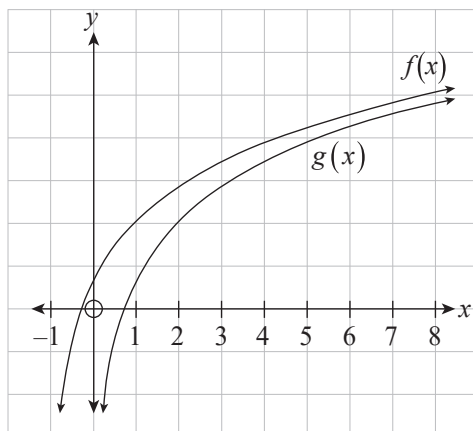


Figure 10

(c) Describe the relationship between the graphs of $y = f(x)$ and $y = g(x)$.

$g(x)$ is equivalent to $f(x)$ translated 1 unit to the right (or $f(x)$ is $g(x)$ translated 1 unit left/negative direction)	<u>key ideas</u> • name of transform • direction • magnitude (1 mark)
---	--

(d) The exact value of A (i.e. the area bounded by the graph of $y = f(x)$, the x -axis, and the vertical lines at $x = 1$ and $x = 7$) can be calculated using a definite integral.

Use part (b)(ii) and your answer to part (c) to determine the exact value of A . Express your answer in the form $p \ln 2 + q$, where p and q are integers.

from (c) $\int_1^7 f(x) dx = \int_2^8 g(x) dx$ $= \int_2^8 \ln(2x^2) dx$ $= \left[x(\ln(2x^2) - 2) \right]_2^8$ from (b)(ii) $= (8 \ln(2 \times 64) - 2) - (2 \ln(2 \times 4) - 2)$ $= 8 \ln 128 - 16 - 2 \ln 8 + 4$ $= 8 \ln 2^7 - 2 \ln 2^3 - 16 + 4$ $= 56 \ln 2 - 6 \ln 2 - 12$ $= 50 \ln 2 - 12$	• Part (c) allows us to switch from $f(x)$ to the easier $g(x)$ • Sub into the (b)(ii) result. • use log laws and index values to simplify to the form required.
--	--

(4 marks)