South Australian Certificate of Education

1

Mathematical Methods
2024

Question booklet 1

- Questions 1 to 6 (53 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on page 14 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

Examination information

Materials
- Question booklet 1
- Question booklet 2
- Formula sheet
- SACE registration number label

Instructions
- Show appropriate working and steps of logic in the question booklets
- State all answers correct to three significant figures, unless otherwise instructed
- Use black or blue pen
- You may use a sharp dark pencil for diagrams and graphical representations

Total time: 130 minutes
Total marks: 100

The SACE Board of South Australia acknowledges that this examination was created on Kaurna Land. We acknowledge First Nations Elders, parents, families, and communities as the first educators of their children, and we recognise and value the cultures and strengths that First Nations students bring to the classroom. We respect the unique connection and relationship that First Nations peoples have to Country, and their ever-enduring cultural heritage.

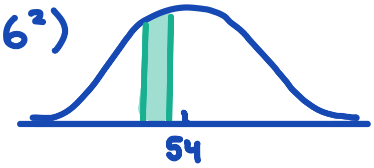
Attach your SACE registration number label here	**Graphics calculator** 1. Brand ______ Model ______ 2. Brand ______ Model ______

Government of South Australia

Question 1 (9 marks)

Let X be a normally distributed random variable with a mean of $\mu = 54$ and a standard deviation of $\sigma = 6$.

$X \sim N(54, 6^2)$

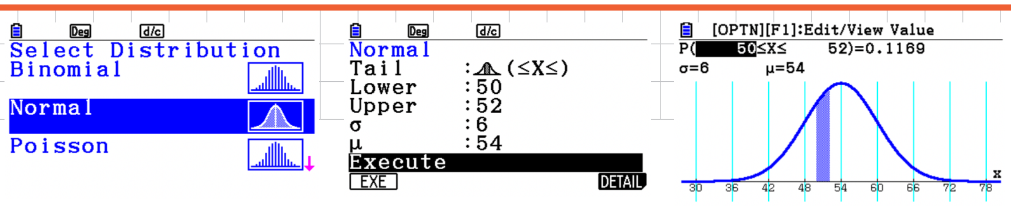


(a) Determine

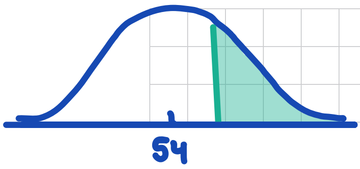
(i) $\Pr(50 \le X \le 52)$.

$\Pr(50 \le X \le 52) = 0.117$

Using the distribution app



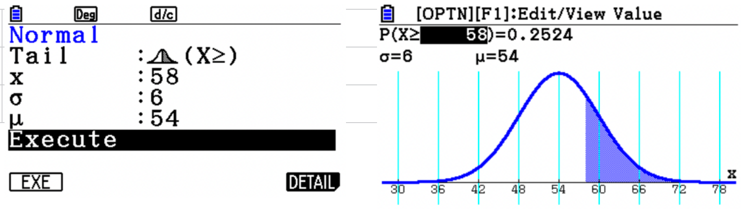
(ii) $\Pr(X \ge 58)$.



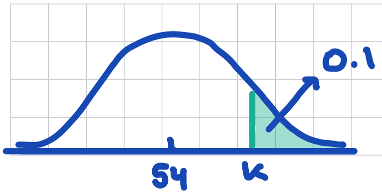
$\Pr(X \geq 58) = 0.252$

Click exit & change tail direction using F3.

Note can also change the set up using OPTN F4 & edit (F1)



(b) Given that $\Pr(X \ge k) = 0.10$, determine the value of k.



$\Pr(X \geq 61.7) = 0.1$

$\therefore k = 61.7$

(1 mark)

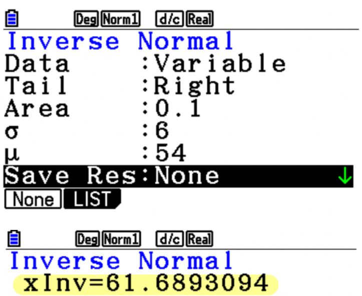
Use statistics app:

DIST F5

Norm F1

InvN F3

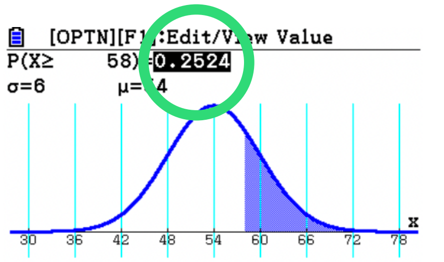
* Note: the above 2 calculations can be done in the statistics app.



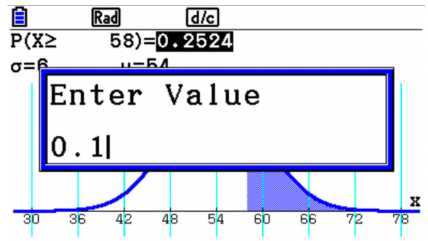
or

Using distⁿ app:

From the prev screen: use the arrows to highlight the probability



Type in 0.1
Press exe



Observe value of k such that $P(X \geq k) = 0.1$

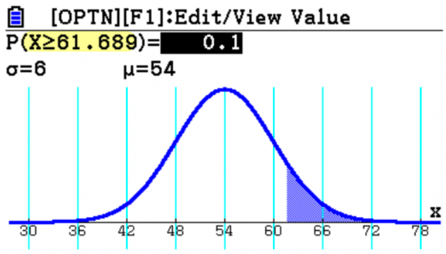


Figure 1 shows the graph of the probability density function for the random variable X. The dashes on the x-axis show increments of one standard deviation.

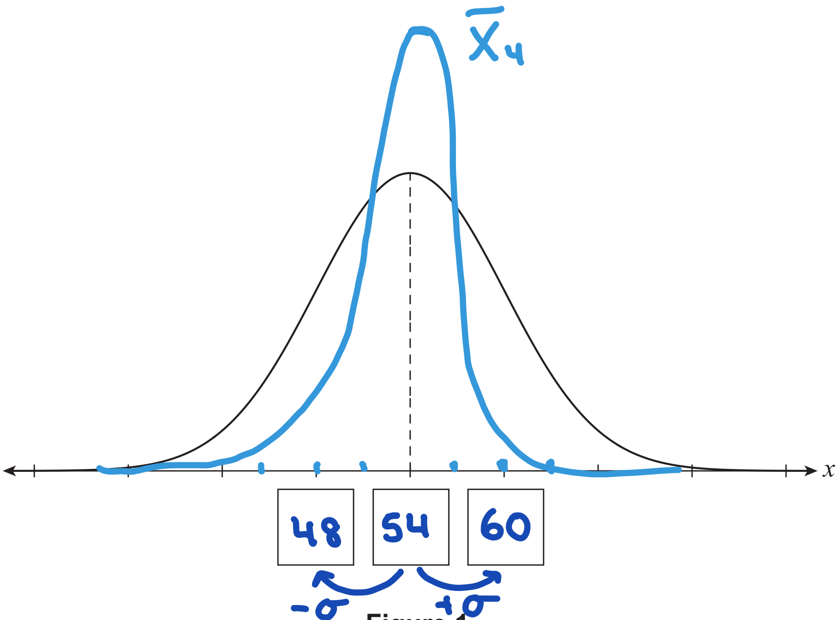


Figure 1

(c) *On Figure 1*, write a number in each box to provide a scale for the x-axis.

don't skim past this question!!

(2 marks)

(d) Let $\bar{X}_4$ be the random variable representing the sample mean of 4 independent observations of X.

(i) State the mean and standard deviation of $\bar{X}_4$.

$\mu_{\bar{X}_4} = 54$ $\sigma_{\bar{X}_4} = \frac{6}{\sqrt{4}} = 3$

(2 marks)

(ii) *On Figure 1 above*, sketch the probability density function for the random variable $\bar{X}_4$.

(2 marks)

* Same mean
* Smaller standard deviation

$\Rightarrow$ probability density function will be skinnier and taller to ensure area under the curve is 1.

Question 2 (9 marks)

Car A and Car B are toy cars powered by compressed carbon dioxide. After they are released from a starting point, the cars travel only forwards in a straight line, and are observed for exactly 2 seconds.

This image has been removed for copyright reasons.

Source: adapted from © LEAP Australia | www.computationalfluiddynamics.com.au

Car A's velocity t seconds after it is released, $v_A(t)$, can be modelled using the function

$$v_A(t) = 3e^{1.5t} - 3$$

where $v_A(t)$ is measured in metres per second and t is measured in seconds for $0 \le t \le 2$.

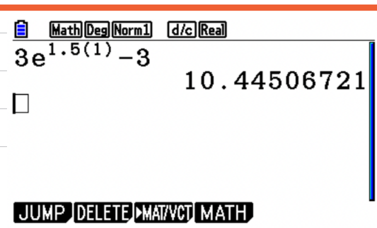
(a) Determine Car A's velocity 1 second after it is released.

$V_A(1) = 3e^{1.5(1)} - 3$

$= 10.4 \ \underline{m/s}$

Units!!

In the run mode:



(b) Let $s_A(t)$ be a function that models the displacement of Car A from the starting point t seconds after it is released.

(i) Show that $s_A(t) = 2e^{1.5t} - 3t - 2$. Assume that $s_A(0) = 0$.

$S_A(t) = \int V_A(t)\, dt$

$S_A(t) = \int 3e^{1.5t} - 3 \ dt$

$= \dfrac{3}{1.5} e^{1.5t} - 3t + C$

$\int e^{at+b} dt = \dfrac{1}{a} e^{at+b} + c$

$= 2e^{1.5t} - 3t + c$

Since $S_A(0) = 0$,

$0 = 2e^0 - 3(0) + c$

$\Rightarrow c = -2$

$\therefore S_A(t) = 2e^{1.5t} - 3t - 2$

(2 marks)

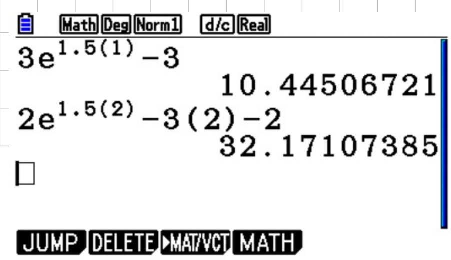
(ii) Hence or otherwise, determine the displacement of Car A from the starting point 2 seconds after it is released.

$S_A(2) = 2e^{1.5(2)} - 3(2) - 2$

$= 32.2 \ \underline{m}$

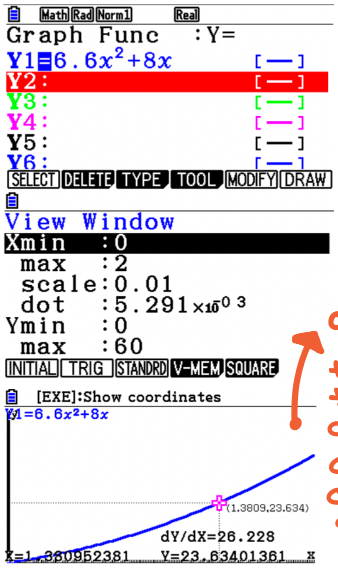
units!

In the run mode:



The graph of $y = v_A(t)$ is shown in Figure 2.

Enter $V_B(t)$ into the graph app and set the v-window (Shift v-window F4) as in Figure 2



click Trace (F1) to see the coordinates of a point on the graph

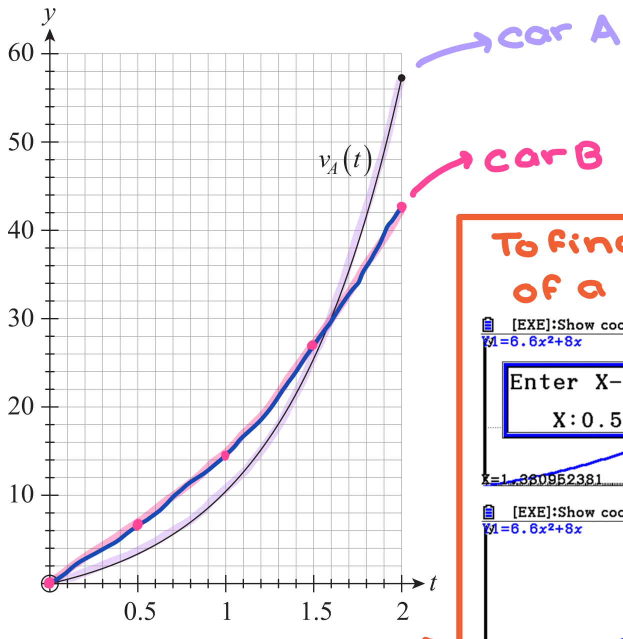
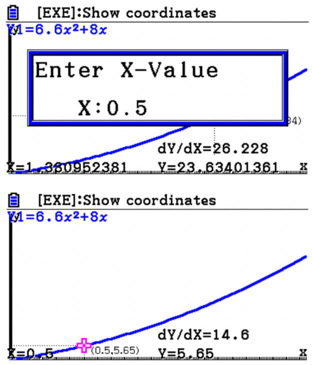


Figure 2

To find the coordinates of a specific point:



Type the x-coordinate of the point

click exe

Car B's velocity t seconds after it is released, $v_B(t)$, can be modelled using the function

$$v_B(t) = 6.6t^2 + 8t$$

where $v_B(t)$ is measured in metres per second and t is measured in seconds for $0 \le t \le 2$.

(c) *On Figure 2 above*, sketch the graph of $y = v_B(t)$.

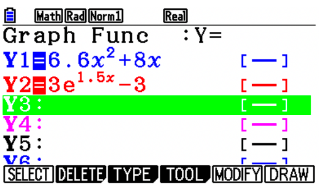
easy to miss!

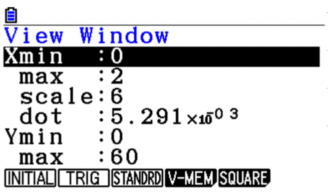
* use graph app to easily access points for plotting by hand.

(2 marks)

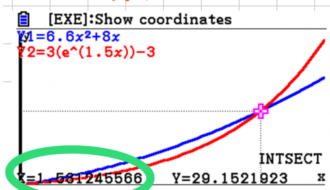
(d) For what values of t was the velocity of Car A greater than the velocity of Car B?

Plot V_A & V_B in graph app using v-window from figure 2





Use F5 G solv, F5 Intsect to find intersection. use arrows to move between intersection points



(1 mark)

$1.58 \le t \le 2$

(e) Which car, Car A or Car B, do the models suggest has travelled the furthest during the 2 seconds that they are observed? Justify your answer.

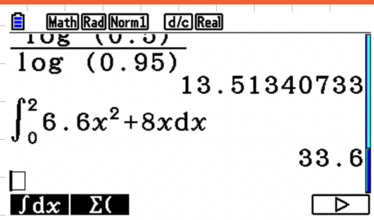
Given the cars only travel forwards, $S_A(t)$ & $S_B(t)$ represent the distance travelled:

$S_A(2) = 32.2\text{ m}$

$S_B(2) = \int_0^2 6.6t^2 + 8t$

$= 33.6$

Math F4
▷ F6
$\int dx$ F1



(2 marks)

$\therefore$ Car B travelled the furthest.

Question 3 (6 marks)

Consider the function $f(x) = \ln(x+3)$ where $x > -3$. A graph of $y = f(x)$ is shown in Figure 3.

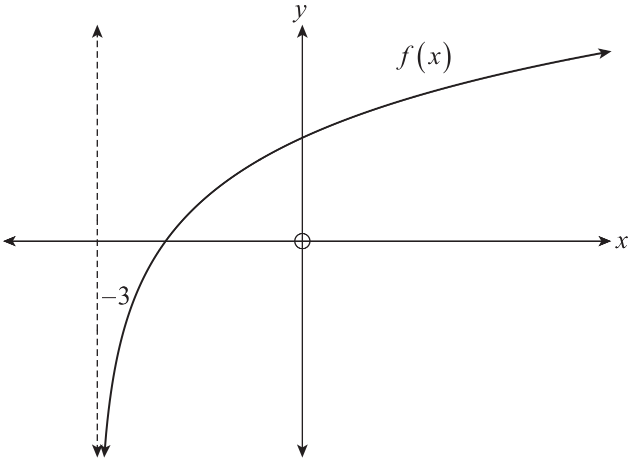


Figure 3

(a) Determine $f'(x)$.

$f'(x) = \dfrac{1}{x+3}$

$\dfrac{d}{dx}\left[\ln(f(x))\right] = \dfrac{f'(x)}{f(x)}$

(1 mark)

(b) Calculate $f'(0)$.

$f'(0) = \dfrac{1}{0+3} = \dfrac{1}{3}$

(1 mark)

Let $g(x) = b\ln(x+3)$, where $x > -3$ and b is a positive constant.

Figure 4 shows the graph of $y = f'(x)$ and $y = g'(x)$. The y-intercepts of each graph are labelled A and B respectively.

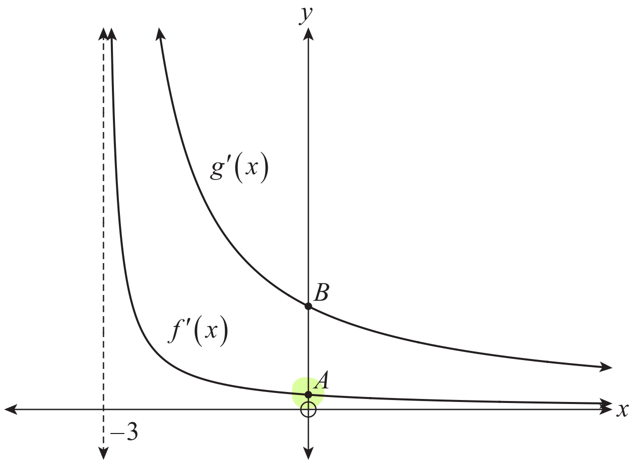


Figure 4

(c) The vertical distance between points A and B is 2 units. Determine the value of b.

point $A \equiv f'(x = 0) \Rightarrow A = (0, \tfrac{1}{3})$

Given A & B are 2 units apart, $B = (0, 2\tfrac{1}{3})$

B is a point on $g'(x)$ ∴ need to find $g'(x)$ → $g'(x) = \dfrac{b}{x+3}$

sub in point B

$2\tfrac{1}{3} = \dfrac{b}{0+3}$

$\therefore b = 3(2\tfrac{1}{3})$

$b = 7$

(3 marks)

(d) Hence or otherwise, state the transformation applied to $f(x)$ to obtain $g(x)$.

∴ vertical dilation by a factor of 7

(1 mark)

Question 4 (12 marks)

$x = 23, n = 100$

(a) Consider a scenario where a random sample size of 100 is taken and 23 successes are observed.

(i) Fill in the boxes below to form a 95% confidence interval for the population proportion of successes, p.

$$\boxed{0.148} \le p \le \boxed{0.312}$$

(1 mark)

(ii) Does the confidence interval calculated in part (a)(i) *contradict* the statement that $p = 0.35$ with 95% confidence? Tick the appropriate box to indicate your answer.

☑ Yes ☐ No (1 mark)

(iii) Justify your answer to part (a)(ii).

It contradicts the statement since 0.35 is not included in the interval

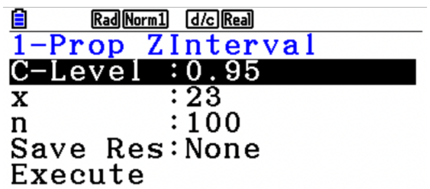
(1 mark)

In the statistics app:

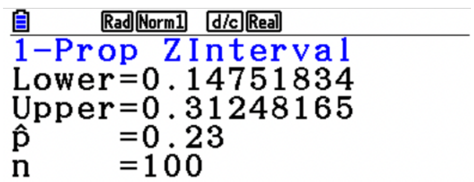
Intr F4

Z F1

1- Prop F1



Press exe



(b) A mathematician is demonstrating that not all calculated confidence intervals will contain the population proportion being investigated.

The mathematician uses a population that is known to have a population proportion of 0.35. They take 50 independent, randomly generated samples of size 100 and calculate 50 independent 95% confidence intervals.

Figure 5 shows these confidence intervals as vertical lines, where the vertical position of the endpoints of each line represents the limits of each confidence interval. Each confidence interval has been assigned a label from 1 to 50.

The population proportion of 0.35 is represented as a horizontal dashed line.

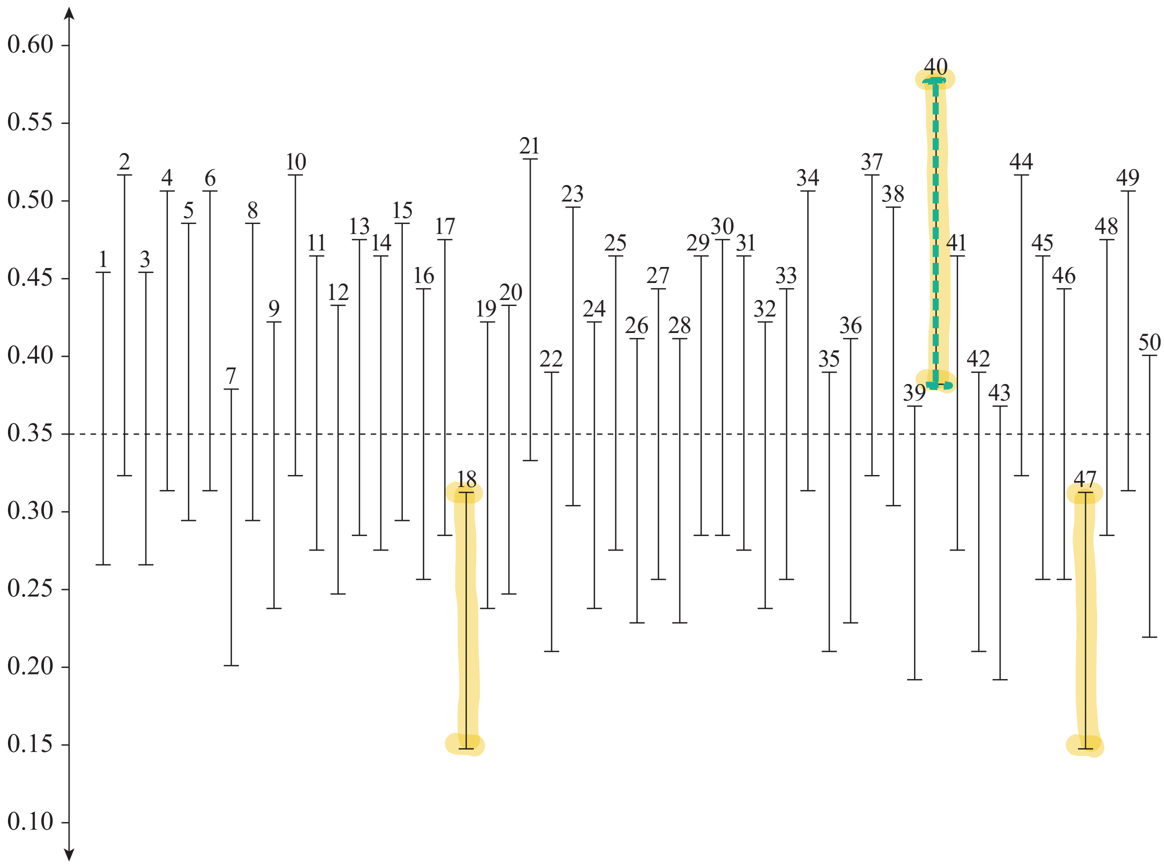


Figure 5

(i) Identify the assigned labels of the *three* confidence intervals that *do not* contain the population proportion.

18, 40 and 47.

(1 mark)

(ii) Identify the assigned label(s) of the confidence interval(s) that could be used to support a claim that the population proportion is greater than 0.35.

40

(1 mark)

PLEASE TURN OVER

When a 95% confidence interval for a population proportion is to be calculated using a random sample of sufficient size, there is a probability of 0.05 that the confidence interval *will not* contain the population proportion.

→ $p = 0.05$

(c) Let X be the number of 95% confidence intervals that *will not* contain the population proportion out of 100 independent 95% confidence intervals to be calculated.

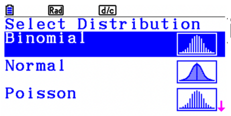
→ num trials $= 100$

(i) Assuming that X can be modelled using a binomial distribution, fill in the boxes below to show the value of the parameters of X.

$$X \sim B\left(\boxed{100}, \boxed{0.05}\right)$$

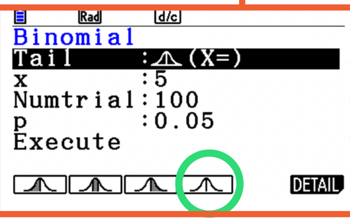
(2 marks)

In the distribution app. Select binomial



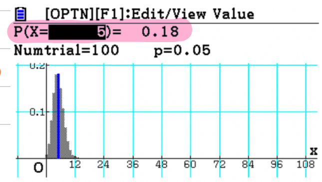
(ii) If 100 new, independent 95% confidence intervals are to be calculated, determine the probability of observing:

(1) exactly five 95% confidence intervals that *will not* contain the population proportion.



Select the correct tail & enter values

Click exe →

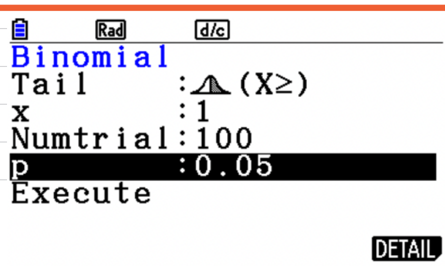


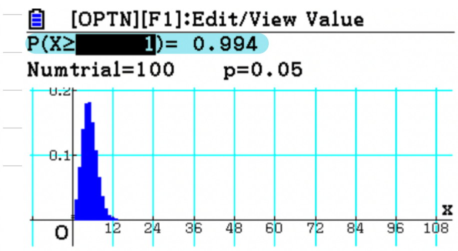
$P(X = 5) = 0.18$

(1 mark)

(2) at least one 95% confidence interval that *will not* contain the population proportion.

$P(X \geq 1) = 0.994$





(d) Determine the minimum number of independent 95% confidence intervals that would need to be calculated to have a greater than 0.5 probability of having at least one 95% confidence interval that *will not* contain the population proportion.

Need to find n such that $Pr(X \geq 1) > 0.5$

$Pr(X \geq 1) = 1 - P(X = 0) > 0.5$

$\therefore\ 1 - P(X=0) > 0.5$

Remember: X is binomial

$P(X=0) < 0.5$

$\binom{n}{0}(0.05)^0(0.95)^n < 0.5$

$\therefore\ 0.95^n < 0.5$

$n \ln(0.95) < \ln(0.5)$

$n > \dfrac{\ln(0.5)}{\ln(0.95)}$

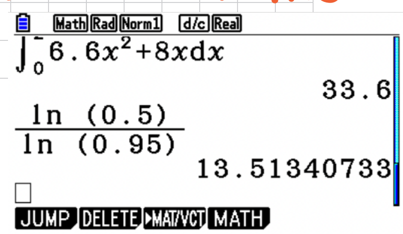
$\ln(0.95) < 0$ ⇒ inequality flips.

$n > 13.5$

Note: could use dist^n app & trial & error.

$\therefore$ The minimum number of confidence intervals is 14.

In the run mode:



Question 5 (5 marks)

Consider the function $f(x)=\frac{x}{x-2}$, where $x \neq 2$.

(a) Use first principles to find $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

don't forget the limit

$$= \lim_{h\to 0} \left(\frac{x+h}{x+h-2} - \frac{x}{x-2}\right) \times \frac{1}{h}$$

$$= \lim_{h\to 0} \left(\frac{(x+h)(x-2) - x(x+h-2)}{(x+h-2)(x-2)}\right) \times \frac{1}{h}$$

$$= \lim_{h\to 0} \left(\frac{\cancel{x^2} - \cancel{2x} + \cancel{hx} - 2h - \cancel{x^2} - \cancel{xh} + \cancel{2x}}{(x+h-2)(x-2)}\right) \times \frac{1}{h}$$

$$= \lim_{h\to 0} \left(\frac{-2\cancel{h}}{(x+h-2)(x-2)}\right) \times \frac{1}{\cancel{h}}$$

$$= \lim_{h\to 0} \left(\frac{-2}{(x+h-2)(x-2)}\right)$$

$$= \frac{-2}{(x-2)^2} \qquad x \neq 2$$

it's easy to check your answer with a quick differentiation:

$$f'(x) = \frac{1(x-2) - x(1)}{(x-2)^2}$$

$$= \frac{-2}{(x-2)^2}$$

(4 marks)

(b) Hence or otherwise, state the slope of the tangent to $f(x)$ when $x = 6$.

$$f'(6) = \frac{-2}{(6-2)^2} = \frac{-2}{4^2} = \frac{-2}{16} = -\frac{1}{8}$$

(1 mark)

Question 6 (12 marks)

A company produces bags of sand which are sold with a labelled net weight of 400 grams. The machine that fills these bags has a stopping mechanism that is designed to stop adding sand when the net weight of the sand in a bag first reaches 400 grams. However, sand often escapes the stopping mechanism, resulting in bags containing excess sand.

The net weight, in grams, of the sand in the bags, X, can be modelled by the probability density function

$$f(x) = \frac{k}{x}$$

where $400 \le x \le 440$ and k is a positive constant.

Source: adapted from
© klyaksun | vecteezy.com

(a) Use an algebraic approach to show that $k = \dfrac{1}{\ln 1.1}$.

$$\int_{400}^{440} \frac{k}{x}\, dx = 1$$

{because f(x) is probability density f^n}

$$\Big[k \ln(x) \Big]_{400}^{440} = 1$$

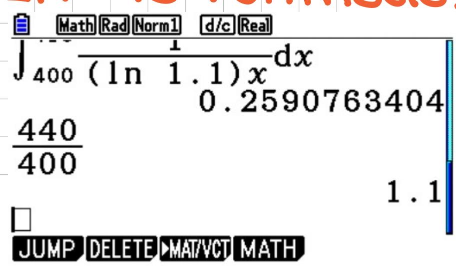
$\ln(A) - \ln(B) = \ln\left(\frac{A}{B}\right)$

$$k\ln(440) - k\ln(400) = 1$$

$$k\left(\ln\left(\frac{440}{400}\right)\right) = 1$$

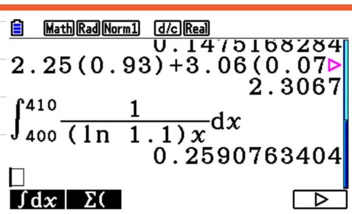
$$\therefore k = \frac{1}{\ln(1.1)}$$

In the run mode:



(b) (i) Calculate $\displaystyle\int_{400}^{410} \frac{1}{(\ln 1.1)x}\, dx$.

In the run mode
MATH F4
▷ F6
∫dx F1



$$\therefore \int_{400}^{410} \frac{1}{(\ln(1.1)x)}\, dx$$

$= 0.259$ (1 mark)

(ii) Interpret your answer to part (b)(i) in the context of the problem.

The probability that one randomly chosen bag weighs between 400 & 410 grams is 0.2591.

(1 mark)

(c) Show that the mean of X is $\mu_X = \dfrac{40}{\ln 1.1}$ grams.

* The question asks us to show $\mu_x = \dfrac{40}{\ln(1.1)}$.
The calculator will not present a exact solution (only a decimal approximation) so ensure to do by hand!!

$$\mu_x = \int_{400}^{440} \cancel{x}\left(\frac{1}{\ln(1.1)\cancel{x}}\right) dx$$

$$= \int_{400}^{440} \frac{1}{\ln(1.1)}\, dx$$

$$= \left[\frac{1}{\ln(1.1)}x\right]_{400}^{440}$$

$$= \frac{440}{\ln(1.1)} - \frac{400}{\ln(1.1)}$$

$$= \frac{40}{\ln(1.1)}$$

(3 marks)

(d) The company alters the stopping mechanism in order to reduce the amount of excess sand in the bags.

After the stopping mechanism has been altered, they take a random sample of 25 bags of sand and find that the mean net weight of the sand in these bags is 417 grams. Assume that the standard deviation is $\sigma_X = 11.5$ grams and the sample size is large enough for the central limit theorem to apply.

(i) Calculate the 99% confidence interval for the mean net weight of the sand in the bags after the stopping mechanism was altered.

In the Statistics app: INTR F4, Z F1, 1 sample F1

```
1-Sample ZInterval
Data    :Variable
C-Level :0.99
σ       :11.5
x̄       :417
n       :25
Save Res:None
None LIST
```

```
1-Sample ZInterval
Lower=411.075593
Upper=422.924407
x̄     =417
n     =25
```

(2 marks)

$411 \leq \mu_x \leq 423$

(ii) The company claims that after the stopping mechanism was altered, the mean net weight of the sand in the bags has reduced. Does the confidence interval calculated in part (d)(i) support this claim with 99% confidence? Justify your answer.

No, since the interval includes the original mean of $\dfrac{40}{\ln(1.1)} \approx 420\text{g}$.

(2 marks)

South Australian Certificate of Education

Mathematical Methods

2024

Question booklet 2

- Questions 7 to 10 (47 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on pages 7, 10, and 14 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

2

Copy the information from your SACE label here

SEQ	FIGURES	CHECK LETTER	BIN

Graphics calculator

1. Brand ____________

   Model ____________

2. Brand ____________

   Model ____________

Government of South Australia

Question 7 (10 marks)

Consider the function $f(x)=\frac{1}{x^2}$. $= x^{-2}$

(a) Use an algebraic approach to determine the equation of the tangent to $f(x)$ at $x=1$.

Slope of tangent $\Rightarrow f'(1)$

$f'(x) = -2x^{-3}$

$f'(1) = -2(1)^{-3} = -2$

tangents have eqⁿ of form $y=mx+c$

$m=-2$ will pass thru $(1, 1/1^2) = (1,1)$

$\therefore$ sub in $(1,1)$ to find c

$\therefore 1 = -2(1)+c$

$c = 1+2$

$c = 3$

$\therefore$ eqⁿ of tangent at $f(1)$ is $y=-2x+3$

(3 marks)

(b) Determine the x-intercept of this tangent.

x int $\Rightarrow y=0$

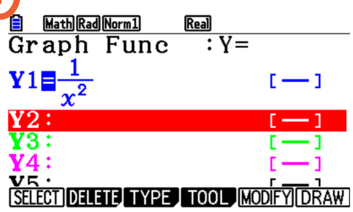
$0 = -2x+3$

$-3 = -2x$ $x = 3/2$

(1 mark)

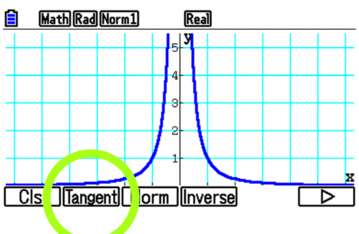
Check your answer to part a on g-calc

1



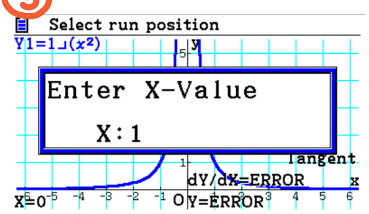
In the graph app:
Enter f(x)
Press F6 to draw

2



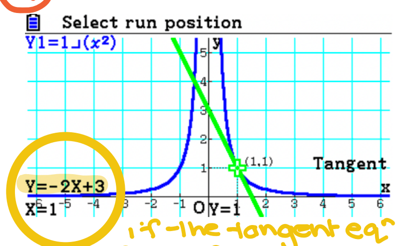
Press Sketch F4 Tangent F2

3



Press 1 to find eqⁿ of tangent at $x=1$.

4



Press exe twice to get solution.

if the tangent eqⁿ does not show:
Shift set up
Press the down arrow until you find derivative
click F1 ON & try again

Consider the function $g(x) = \frac{1}{x^2+k}$, where k is a non-negative constant. The x-intercept of the tangent to $g(x)$ at $x = 1$ is shown in Table 1 for various values of k.

If you are stuck add $k = 0$ (Q7a & b) to table to help you see the pattern

Table 1

k	0	1	2	3	4
$g(x)$	$\frac{1}{x^2}$	$\frac{1}{x^2+1}$	$\frac{1}{x^2+2}$	$\frac{1}{x^2+3}$	$\frac{1}{x^2+4}$
x-intercept of the tangent to $g(x)$ at $x = 1$	$\left(\frac{3}{2}, 0\right)$	$(2, 0)$ $\frac{4}{2}$	$\left(\frac{5}{2}, 0\right)$	$(3, 0)$ $\frac{6}{2}$	$\left(\frac{7}{2}, 0\right)$

(c) Make a conjecture about the x-intercept of the tangent to $g(x) = \frac{1}{x^2+k}$ at $x = 1$.

$$\left(\frac{k+3}{2}, 0\right)$$

(1 mark)

(d) Prove or disprove the conjecture that you made in part (c).

Want to show x int of tangent at $x = 1$ is $\frac{k+3}{2}$

1. Find $g'(1)$

$$g(x) = (x^2+k)^{-1}$$

$$g'(x) = -1(x^2+k)^{-2}(2x)$$

$$= \frac{-2x}{(x^2+k)^2} \qquad \therefore g'(1) = \frac{-2}{(1+k)^2}$$

2. Find eq^n of tangent

$$y = \frac{-2}{(k+1)^2}x + c$$

passes thru pt $\left(1, \frac{1}{1+k}\right)$

$$\frac{1}{1+k} = \frac{-2}{(k+1)^2} + c$$

$$\therefore c = \frac{1}{1+k} + \frac{2}{(k+1)^2}$$

$$\therefore c = \frac{k+1+2}{(k+1)^2}$$

$$= \frac{k+3}{(k+1)^2}$$

$$y = \frac{-2}{(k+1)^2}x + \frac{k+3}{(k+1)^2}$$

3. Find x int ($y = 0$)

$$0 = \frac{-2}{(k+1)^2}x + \frac{k+3}{(k+1)^2}$$

$$\frac{k+3}{(k+1)^2} = \frac{2}{(k+1)^2}x$$

$$x = \frac{k+3}{2} \text{ as req.}$$

(5 marks)

Question 8 (12 marks)

Park-o-Rama is a multi-storey 24-hour parking garage. On a day in 2024, t hours after midnight, the rate at which vehicles:

- enter Park-o-Rama can be modelled by the function $E(t)$
- leave Park-o-Rama can be modelled by the function $L(t)$.

Both $E(t)$ and $L(t)$ are measured in vehicles per hour for $0 \leq t \leq 24$, and Park-o-Rama contains no vehicles when $t = 0$.

Graphs of $y = E(t)$ and $y = L(t)$ are shown in Figure 6.

Source: adapted from © s99 | iStock.com

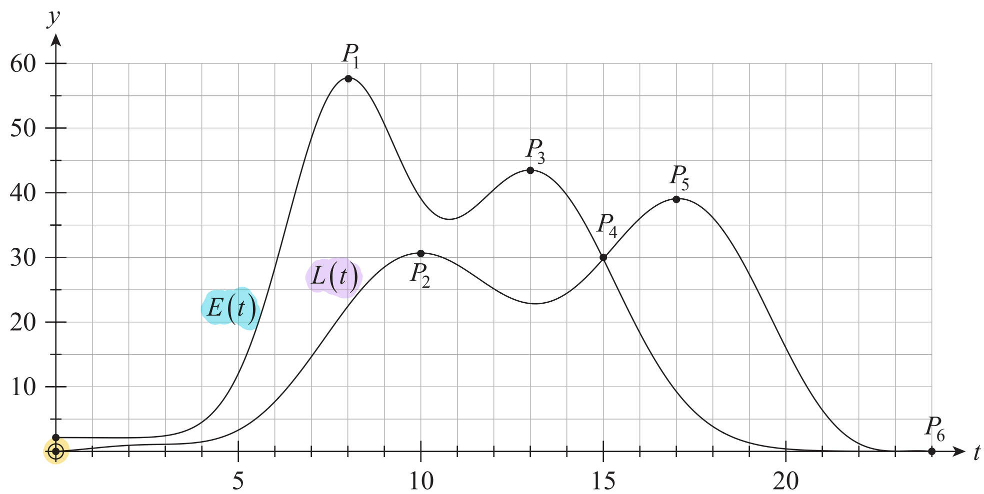


Figure 6

The t-coordinates of the labelled points and the values of various integral expressions are shown in Table 2. Assume that P_1, P_2, P_3, and P_5 are stationary points.

Table 2

Point P_i	P_1	P_2	P_3	P_4	P_5	P_6
t_i, the t-coordinate of P_i	8	10	13	15	17	24
$\int_0^{t_i} E(t)\,dt$	130	230	347	429	462	471
$\int_0^{t_i} L(t)\,dt$	40	96	177	227	298	406

(a) (i) It is known that $E(9) = 50.6$. State the meaning of this result in the context of the problem.

E(t) model the rate at which cars enter the car park.
$\therefore$ $E(9) = 50.6$ means cars enter at a rate of 50.6 vehicles per hour at 9 hours after midnight.

(2 marks)

(ii) Use the information given on page 4 to determine:

(1) how many hours after midnight vehicles were *leaving* Park-o-Rama at the maximum rate.

max rate of vehicles leaving occurs at P_5 i.e. when $t = 17$.
$\therefore$ 17 hours after midnight

(1 mark)

(2) how many vehicles *entered* Park-o-Rama during the 24-hour period.

$$\int_0^{24} E(t)\,dt = 471$$

(1 mark)

17 to 24 hours

(3) how many vehicles *left* Park-o-Rama during the final 7 hours of the day.

$$\int_{17}^{24} L(t)\,dt = \int_0^{24} L(t)\,dt - \int_0^{17} L(t)\,dt$$

$$= 406 - 298$$

$$= 108$$

(1 mark)

(iii) The maximum number of vehicles in Park-o-Rama was 202. How many hours after midnight did this occur?

# of vehicles at time t $= \int_0^t E(t)\,dt - \int_0^t L(t)\,dt = 202$

(1 mark)

from table 2:

$$\int_0^{15} E(t)\,dt - \int_0^{15} L(t)\,dt$$

$$= 429 - 227$$

$$= 202$$

$\therefore$ $t = 15$. The maximum number of vehicles occurs 15 hours after midnight.

 PLEASE TURN OVER

(b) Recently, more people have started driving electric vehicles (EVs).

In 2024, one study found that:

- non-EVs have an average weight of 2.25 tonnes
- EVs are 36% heavier than non-EVs
- 7% of vehicles are EVs.

(i) Use the information given above to complete Table 3.

EV's are 36% heavier than non EVs
$\therefore 2.25 \times 1.36$
$= 3.06$ T

Table 3

	Non-EVs	*EVs*
Weight (tonnes)	2.25	3.06
Proportion of vehicle type	0.93	0.07

if 7% of vehicles, 93% are non ev.

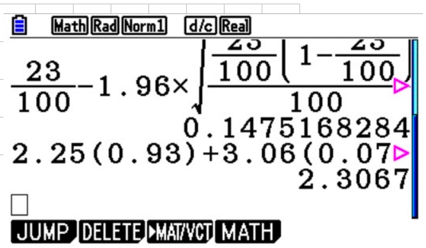
(2 marks)

Let X represent the weight in tonnes of a randomly selected vehicle in 2024.

(ii) Use the information in Table 3 to show that X has a mean of $\mu_X = 2.31$ tonnes (correct to three significant figures).

$\mu_X = \sum x\, p_x$
$= 2.25(0.93) + 3.06(0.07)$
$= 2.31$

In the run mode:



The engineers working for Park-o-Rama are studying the day considered on page 4. They would be concerned if the total weight of vehicles in Park-o-Rama exceeds 475 tonnes with a probability greater than 0.005 at any point during the day.

(iii) Determine whether the engineers would be concerned.

Assume that your chosen sample size is large enough for the central limit theorem to apply, the vehicles in Park-o-Rama constitute a random sample of vehicles, and that the weights of the vehicles, X, have $\mu_X = 2.31$ and $\sigma_X = 0.207$ tonnes.

We know the max # of cars is 202 however we don't know the mass of the cars.
We know X has $\mu_X = 2.31$ and $\sigma_X = 0.207$.
To see if the engineers are concerned:
calc Pr(the mass of 202 cars > 475 T)

$\bar{x} = \frac{475}{202}$

using: $\bar{X}_{202}$

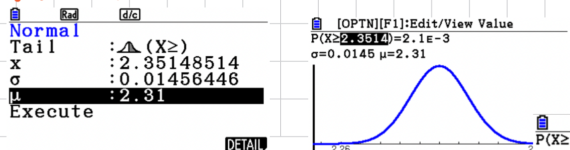
$\mu_{\bar{X}_{202}} = 2.31$

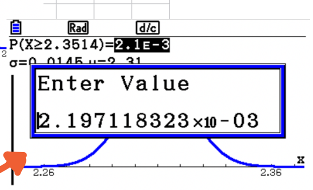
$\sigma_{\bar{X}_{202}} = \frac{0.207}{\sqrt{202}}$

$\therefore P(\bar{X} \geq \bar{x}) = 0.002 < 0.05$

$\therefore$ no concern.

In the distribution app:





Note: To see full sol^n, not just 1 d.p, use right arrow to highlight 3E-3, click optn, edit (F1)

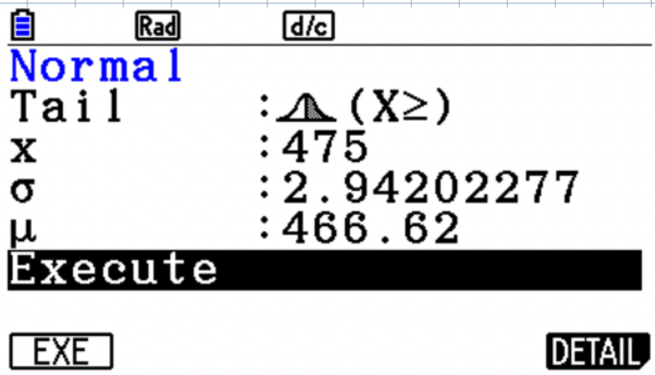
Alternate solution to 8b(iii)

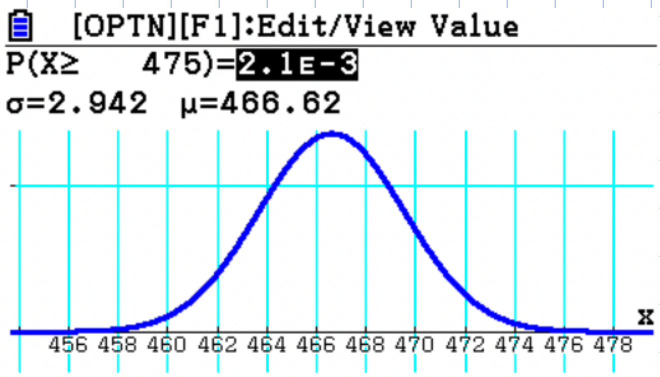
Instead of using the distribution of the sample mean, you could use the distribution of the sample sums.

Let S_{202} be the distribution of sample sums for samples of size 202.

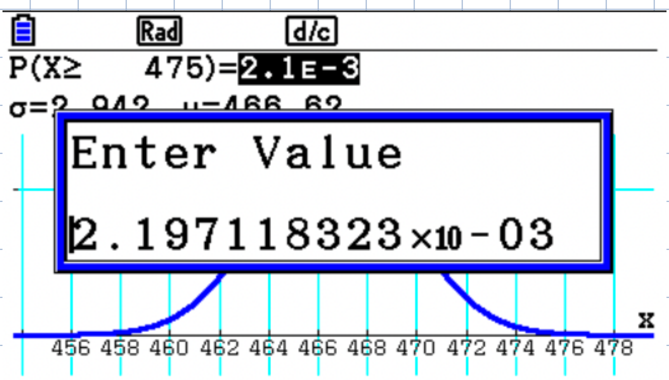
$\therefore S_{202} \sim N(202 \times 2.31,\ 0.207 \times \sqrt{202})$

Using the distribution app:





* When the probabilities are small, you can see the true value by highlighting the 2.1E-3 and press: optn Edit F1



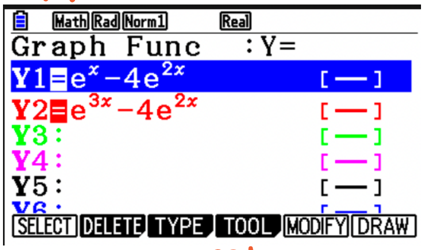
$\therefore P(S_{202} > 475) \approx 0.002 < 0.05$

$\therefore$ no concern

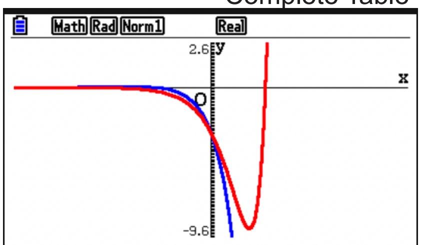
Question 9 (12 marks)

(a) Consider the functions $f(x)=e^{x}-4e^{2x}$ and $g(x)=e^{3x}-4e^{2x}$. Figure 7 shows the graph of $y=f(x)$ and Figure 8 shows the graph of $y=g(x)$ with the stationary point of each function labelled as P and Q respectively.

In the graph app:



* Ensure to set V-window to capture both P and Q



Use G-Solv F5 & max or min to find P & Q. To select which fn, use the down arrow.

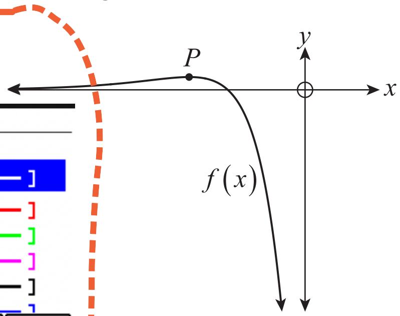


Figure 7

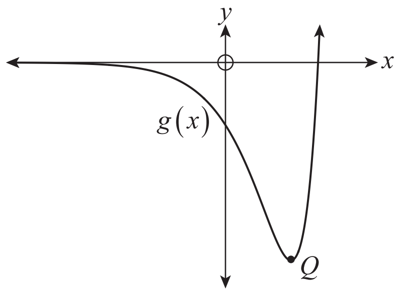


Figure 8

Complete Table 4 below by finding the y-coordinates of P and Q.

Table 4

Point	y-coordinate
P	0.0625
Q	-9.481

(2 marks)

use g-calc as only two marks

(b) Consider the function $y=e^{ax}-4e^{2x}$ where a is a positive constant such that $a\neq 2$.

(i) Show that the x-coordinate of the only stationary point of $y=e^{ax}-4e^{2x}$ is $x=\frac{1}{a-2}\ln\left(\frac{8}{a}\right)$.

Stationary point occurs when $y'(x)=0$

$$y'(x)=ae^{ax}-8e^{2x}=0$$

$$\therefore\ ae^{ax}=8e^{2x}$$

$$\frac{e^{ax}}{e^{2x}}=\frac{8}{a} \qquad \frac{a^b}{a^c}=a^{b-c}$$

$$e^{(a-2)x}=\frac{8}{a}$$

$$\ln\left(e^{(a-2)x}\right)=\ln(8/a) \qquad \ln e^{Ax}=Ax$$

$$(a-2)x=\ln(8/a)$$

$$x=\frac{1}{a-2}\ln\left(\frac{8}{a}\right).$$

(4 marks)

(ii) (1) Show that the y-coordinate of the stationary point of $y = e^{ax} - 4e^{2x}$ is $y = 4\left(\frac{8}{a}\right)^{\frac{2}{a-2}}\left(\frac{2-a}{a}\right)$.

Sub $x = \frac{1}{a-2}\ln\left(\frac{8}{a}\right)$ into $y = e^{ax} - 4e^{2x}$

$$y = e^{ax} - 4e^{2x}$$

$$y = e^{a(1/(a-2)\ln(\frac{8}{a}))} - 4e^{2(1/(a-2)\ln(8/a))}$$

$$y = e^{\ln((\frac{8}{a})^{a/(a-2)})} - 4e^{\ln((\frac{8}{a})^{2/(a-2)})}$$

$$y = \left(\frac{8}{a}\right)^{a/(a-2)} - 4\left(\frac{8}{a}\right)^{2/(a-2)}$$

$$= 4\left(\frac{8}{a}\right)^{2/(a-2)}\left(\frac{2}{a} - 1\right)$$

$$= 4\left(\frac{8}{a}\right)^{2/(a-2)}\left(\frac{2-a}{a}\right)$$

Remove required common factor

$$\frac{\left(\frac{8}{a}\right)^{\frac{a}{a-2}}}{4\left(\frac{8}{a}\right)^{\frac{2}{a-2}}}$$

$$= \frac{1}{4}\left(\frac{8}{a}\right)^{\frac{a-2}{a-2}}$$

$$= \frac{1}{4}\left(\frac{8}{a}\right)$$

$$= \frac{2}{a}$$

(4 marks)

(2) Determine the values of a for which the y-coordinate of the stationary point of the function $y = e^{ax} - 4e^{2x}$ is positive. Justify your answer.

y coordinate : $4\left(\frac{8}{a}\right)^{2/(a-2)}\left(\frac{2-a}{a}\right) > 0$ where $a > 0$ & $a \neq 2$

given $a > 0$:

$$\Rightarrow \frac{8}{a} > 0$$

$$\Rightarrow \left(\frac{8}{a}\right)^{2/(a-2)} > 0$$

Now consider $\left(\frac{2-a}{a}\right)$

We want $\frac{2-a}{a} > 0$

$$\Rightarrow 2 - a > 0$$

$$\Rightarrow -a > -2$$

$$\Rightarrow a < 2$$

$\therefore$ $0 < a < 2$ ensures the y-coordinate of the stationary point of y is positive.

(2 marks)

Question 10 (13 marks)

Consider the function $f(x)=e^{-\cos x}$ for $0 \le x \le 2\pi$ with derivative $f'(x)=e^{-\cos x}\sin x$.
Figure 9 shows the graph of $y=f(x)$ and $y=f'(x)$. The two functions touch at P.

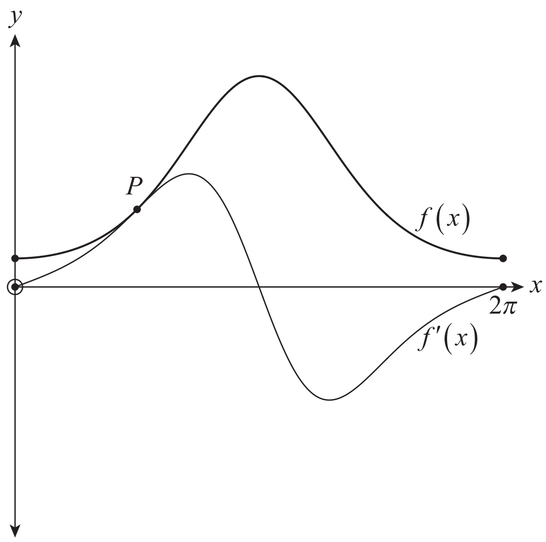


Figure 9

(a) Use an algebraic approach to determine the exact x-coordinate of P.

The point P will have coordinates

on $f(x)$: $(p_1, e^{-\cos(p_1)})$

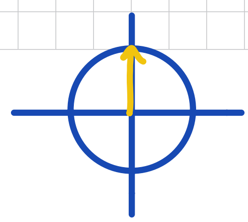
on $f'(x)$: $(p_1, e^{-\cos(p_1)}\sin(p_1))$

$\therefore\ e^{\cancel{-\cos(p_1)}} = e^{\cancel{-\cos(p_1)}}\sin(p_1)$

$1 = \sin(p_1)$

$\therefore\ p_1 = \sin^{-1}(1)$

$p_1 = \frac{\pi}{2}$



(3 marks)

$\therefore$ P will have coordinates

$\left(\frac{\pi}{2}, e^{-\cos(\pi/2)}\right) = \left(\frac{\pi}{2}, 1\right)$

$\cos(\pi/2) = 0$

Recall that $f(x)=e^{-\cos x}$ for $0 \le x \le 2\pi$ with derivative $f'(x)=e^{-\cos x}\sin x$.

(b) (i) Show that $f''(x)=e^{-\cos x}\left(-\cos^2 x+\cos x+1\right)$.

Note that $\cos^2 x+\sin^2 x=1$. $\longrightarrow$ $\sin^2 x = 1-\cos^2 x$

$$f'(x) = e^{-\cos x}(\sin x)$$
$$f''(x) = \sin x\, e^{-\cos x}(\sin x) + e^{-\cos x}(\cos x)$$
$$= e^{-\cos x}(\sin^2 x + \cos x)$$

Sub in $\sin^2 x = 1-\cos^2 x$

$$= e^{-\cos x}(1-\cos^2 x + \cos x)$$
$$= e^{-\cos x}(-\cos^2 x + \cos x + 1)$$

(2 marks)

(ii) Hence, use an algebraic approach to determine the values of x such that $f''(x)=f(x)$.

$$f''(x) = f(x)$$
$$\Rightarrow \cancel{e^{-\cos x}}(-\cos^2 x + \cos x + 1) = \cancel{e^{-\cos x}}$$
$$-\cos^2 x + \cos x + 1 = 1$$
$$-\cos^2 x + \cos x = 0$$
$$\cos x(-\cos x + 1) = 0$$
$$\therefore \cos x = 0 \quad \text{or} \quad -\cos x + 1 = 0$$
$$\Rightarrow \cos x = 1$$

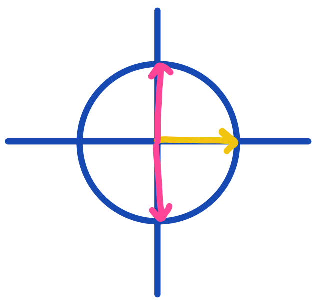
(3 marks)

Since $0 \le x \le 2\pi$:

$x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

or

$x = 0$ or 2π



$\therefore f''(x) = f(x)$ when

$x = 0, \frac{\pi}{2}, \frac{3\pi}{2}$ & 2π. $\{0 \le x \le 2\pi\}$

Figure 10 shows the graph of $y = f'(x)$ and $y = f''(x)$, which intersect when $x = k$ and at one other point. The two distinct regions enclosed between the two functions from $x = k$ to $x = 2\pi$ are denoted as A and B, as shown in Figure 10. The areas of these regions are measured in units squared.

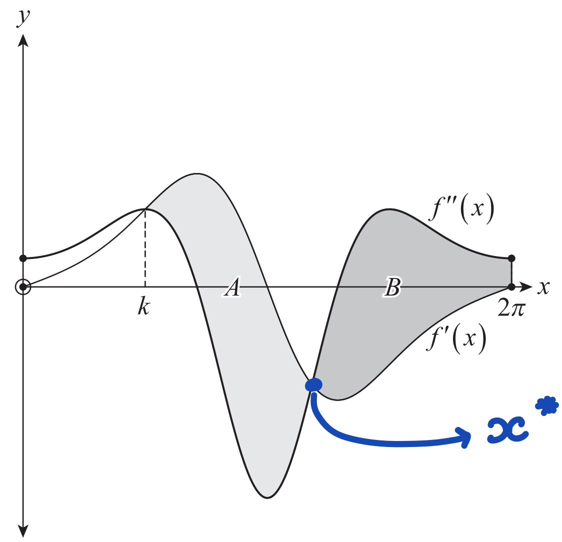


Figure 10

take careful note which questions they refer to. It will help you answer the Question.

(c) Use part (a) and part (b)(ii) to explain why $k = \frac{\pi}{2}$.

In part bii, we showed $f''(x) = f(x)$ when $x = 0, \pi/2, 3\pi/2$ & 2π. In part a, we showed that $f(\pi/2) = f'(\pi/2)$.

$\therefore$ given $f''(\pi/2) = f(\pi/2)$ and $f(\pi/2) = f'(\pi/2)$,
$f''(\pi/2) = f'(\pi/2)$
$f'(x)$ & $f''(x)$ meet at point k
$\therefore k = \pi/2$.

(1 mark)

(d) Use an algebraic approach to determine the exact difference in area between region A and region B.

Diff in area = A − B

$$= \int_{\pi/2}^{x^*} \left(f'(x) - f''(x)\right) dx - \int_{x^*}^{2\pi} \left(f''(x) - f'(x)\right) dx$$

$$= \int_{\pi/2}^{x^*} f'(x)dx - \int_{\pi/2}^{x^*} f''(x)dx - \int_{x^*}^{2\pi} f''(x)dx + \int_{x^*}^{2\pi} f'(x)dx$$

$$= \int_{\pi/2}^{2\pi} f'(x)\, dx - \int_{\pi/2}^{2\pi} f''(x)\, dx$$

see next page for rest of sol^n

(4 marks)

$$= \Big[f(x) \Big]_{\pi/2}^{2\pi} - \Big[f'(x) \Big]_{\pi/2}^{2\pi}$$

$$= \Big[e^{-\cos x} \Big]_{\pi/2}^{2\pi} - \Big[e^{-\cos x} \sin x \Big]_{\pi/2}^{2\pi}$$

$$= [e^{-1} - e^{0}] - [e^{1}(0) - e^{0}(1)]$$

$$= e^{-1} - 1 + 1$$

$$= \frac{1}{e}$$

MATHEMATICAL METHODS FORMULA SHEET

Properties of derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(f(x)\,g(x)\Big) = f'(x)g(x) + f(x)g'(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$$

Quadratic equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Discrete random variables

The mean or expected value of a discrete random variable is:

$$\mu_X = \sum xp(x),$$

where $p(x)$ is the probability function for achieving result x.

The standard deviation of a discrete random variable is:

$$\sigma_X = \sqrt{\sum\left[x - \mu_X\right]^2 p(x)},$$

where μ_X is the expected value and $p(x)$ is the probability function for achieving result x.

Bernoulli distribution

The mean of the Bernoulli distribution is p, and the standard deviation is:

$$\sqrt{p(1-p)}.$$

Binomial distribution

The mean of the binomial distribution is np, and the standard deviation is:

$$\sqrt{np(1-p)},$$

where p is the probability of success in a single Bernoulli trial and n is the number of trials.

The probability of k successes from n trials is:

$$\Pr(X = k) = C_k^n p^k (1-p)^{n-k},$$

where p is the probability of success in the single Bernoulli trial.

Population proportions

The sample proportion is $\hat{p} = \dfrac{X}{n}$,

where a sample of size n is chosen, and X is the number of elements with a given characteristic.

The distribution of a sample proportion has a mean of p and a standard deviation of

$$\sqrt{\frac{p(1-p)}{n}}.$$

The upper and lower limits of a confidence interval for the population proportion are:

$$\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq p \leq \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},$$

where the value of z is determined by the confidence level required.

Continuous random variables

The mean or expected value of a continuous random variable is:

$$\mu_X = \int_{-\infty}^{\infty} x\,f(x)\,\mathrm{d}x,$$

where $f(x)$ is the probability density function.

The standard deviation of a continuous random variable is:

$$\sigma_X = \sqrt{\int_{-\infty}^{\infty}\left[x - \mu_X\right]^2 f(x)\,\mathrm{d}x},$$

where $f(x)$ is the probability density function.

Normal distributions

The probability density function for the normal distribution with mean μ and standard deviation σ is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

All normal distributions can be transformed to the standard normal distribution with $\mu = 0$ and $\sigma = 1$ by:

$$Z = \frac{X - \mu}{\sigma}.$$

Sampling and confidence intervals

If $\overline{x}$ is the sample mean of a sufficiently large sample, and σ is the population standard deviation, then the upper and lower limits of the confidence interval for the population mean are:

$$\overline{x} - z\frac{\sigma}{\sqrt{n}} \leq \mu \leq \overline{x} + z\frac{\sigma}{\sqrt{n}},$$

where the value of z is determined by the confidence level required.