South Australian
Certificate of Education

1

Mathematical Methods
2025

Question booklet 1

- Questions 1 to 6 (50 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on page 14 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

Suggested Solutions in purple

additional detail and notes (not necessarily part of the solution) in pink

Notes on use of Casio Fx-1AU GRAPH in green

Examination information

Materials
- Question booklet 1
- Question booklet 2
- Formula sheet
- SACE registration number label

Instructions
- Show appropriate working and steps of logic in the question booklets
- State all answers correct to three significant figures, unless otherwise instructed
- Use black or blue pen
- You may use a sharp dark pencil for diagrams and graphical representations

Total time: 130 minutes
Total marks: 100

NB: these are not official SACE Board Solutions!

The SACE Board of South Australia acknowledges that this examination was created on Kaurna Land. We acknowledge First Nations Elders, parents, families, and communities as the first educators of their children, and we recognise and value the cultures and strengths that First Nations students bring to the classroom. We respect the unique connection and relationship that First Nations peoples have to Country, and their ever-enduring cultural heritage.

Attach your SACE registration number label here	**Graphics calculator** 1. Brand ______ Model ______ 2. Brand ______ Model ______

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Question 1 (8 marks)

(a) Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following functions. You do not need to simplify your answers.

(i) $y = \ln(7x) - 4\sqrt{x}$.

$$\frac{dy}{dx} = \frac{7}{7x} - 4 \times \frac{1}{2} x^{-1/2}$$

$$\left(= \frac{1}{x} - \frac{2}{\sqrt{x}} \right)$$

(2 marks)

(ii) $y = \left(\frac{2}{x} + \cos x\right)^5$.

$$\frac{dy}{dx} = 5\left(\frac{2}{x} + \cos x\right)^4 \times \left(-2x^{-2} - \sin x\right)$$

(2 marks)

(iii) $y = (2x^3 + 7)(3 + e^{6x})$.

$$\frac{dy}{dx} = (6x^2)(3 + e^{6x}) + (2x^3 + 7)(e^{6x} \times 6)$$

(2 marks)

(b) Determine $\int \left(\frac{7x+4}{x}\right) \mathrm{d}x,\ x > 0$.

$$= \int 7 + \frac{4}{x}\, dx$$

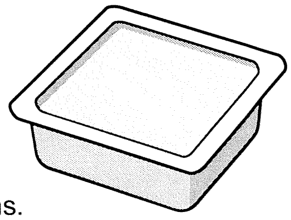
$$= 7x + \ln x + c$$

(or $\ln|x|$ (but not essential, as $x > 0$))

(2 marks)

Question 2 (8 marks)

A company manufactures tofu blocks in two sizes: small blocks with a labelled weight of 400 grams, and large blocks with a labelled weight of 600 grams.

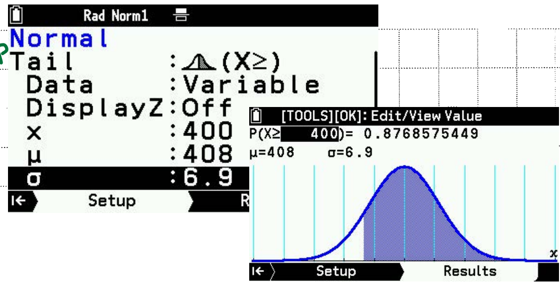


(a) The weights of individual small blocks are known to be normally distributed with a mean of 408 grams and a standard deviation of 6.9 grams.

Determine the probability that the weight of a randomly selected *small* block is *more* than its labelled weight.

0.877

- Distribution App
- Normal
- Enter data
- press Next →



(1 mark)

(b) The weights of individual large blocks are known to be normally distributed with a mean of 624 grams. The company knows that the proportion of large blocks that weigh more than their labelled weight is 0.987.

Using the standard normal distribution (i.e. $Z \sim N(0,1)$), determine the standard deviation of the weights of individual *large* blocks.

← can't use trial and error or SolveN

or using the Calculate app + Catalog

Find InvNormCD via · Catalog
- All ok
- i cos
- then ...

Syntax is ...(tail, prob, μ, σ)

-1 ⇒ left
0 ⇒ centre
1 ⇒ right

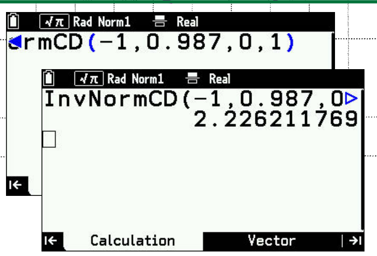
$\Pr(Z \geq z) = 0.987$

$z = -2.2262$ — using Distribution app

$\therefore -2.2262 = \dfrac{600-624}{\sigma}$

$\therefore \sigma = \dfrac{-24}{-2.2262}$

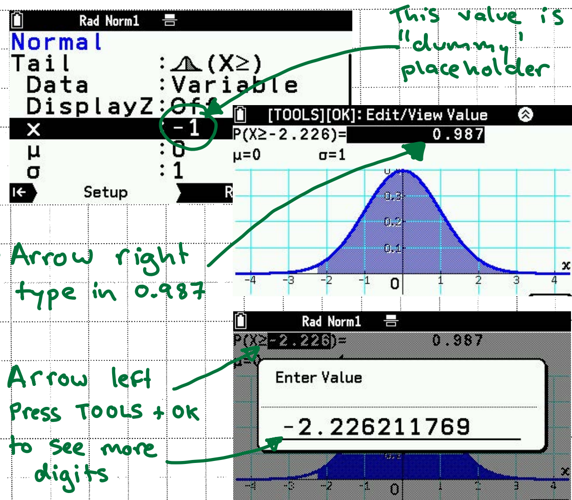
$= 10.8$ (3 sf)



This value is a "dummy" placeholder

- Arrow right
- type in 0.987

- Arrow left
- Press Tools + ok to see more digits



(3 marks)

(c) Which size of block (small or large) is more likely to weigh *less* than its labelled weight?

Small

(as $1-0.877 > 1-0.987$)

(1 mark)

Small tofu blocks are sold in boxes. Each box contains 25 randomly selected small blocks.

(d) (i) Show that the mean and standard deviation for the net weight of a randomly selected box is 10 200 grams and 34.5 grams respectively.

$\mu_{S_{25}} = \mu_S \times 25 = 408 \times 25 = 10200$

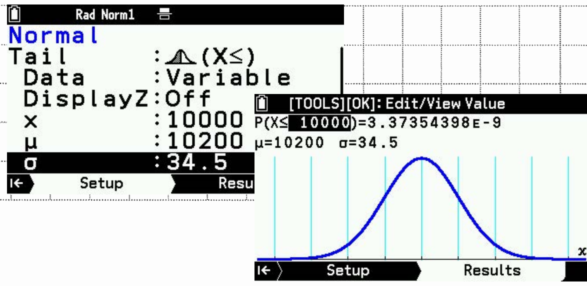
$\sigma_{S_{25}} = \mu_S \times \sqrt{25} = 6.9 \times 5 = 34.5$

(2 marks)

(ii) Hence, determine the probability that the net weight of a randomly selected box is *less* than 10 kilograms (i.e. 10 000 grams).

$Pr(X_{S_{25}} \leq 10000)$

$= 3.37 \times 10^{-9}$



(1 mark)

PLEASE TURN OVER

Question 3 (10 marks)

Consider the function $f(x) = -\frac{1}{10}x^3 - 2x + 20$ where $x \geq 0$. The graph of $y = f(x)$ is shown in Figure 1 along with a shaded region. The shaded region is bounded by $y = f(x)$, the x-axis, the y-axis, and the vertical line $x = 1$.

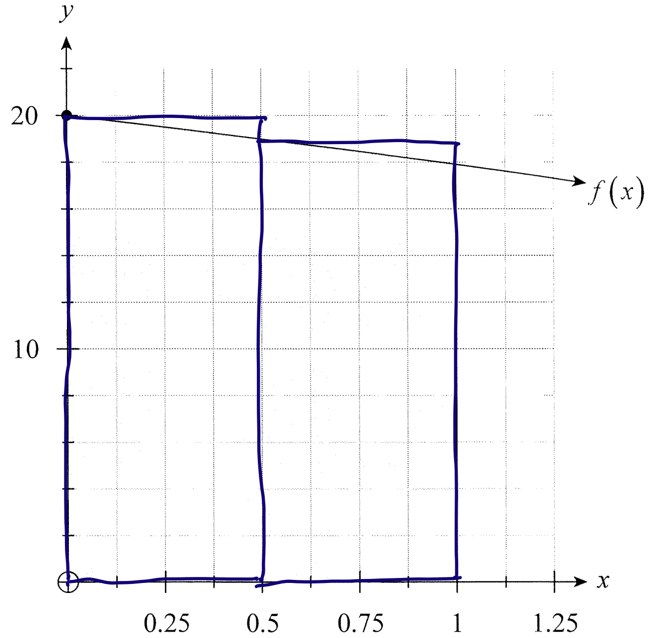


Figure 1

Let U_2 represent an upper estimate of the area of the shaded region using two rectangles of equal width.

(a) On Figure 1, draw two rectangles of equal width that could be used to calculate U_2.

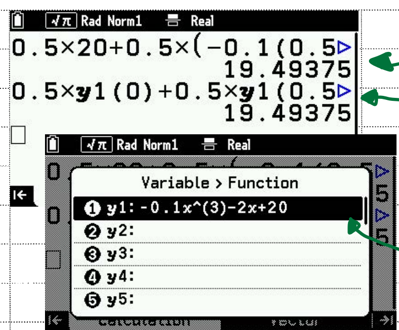
(1 mark)

(b) Calculate the value of U_2.

Can be entered directly

$U_2 = \frac{1}{2} \times f(0) + \frac{1}{2} \times f(0.5)$

$= 19.5$ (3 s.f.)



Can be calculated using a function entered as y1 in the G+T app

to call up "y1" from G+T app press
- VARIABLE
- FUNCTION ②
- y1 ①

(2 marks)

(c) Determine $f''(x)$.

$f'(x) = -0.3x^2 - 2$

$\therefore \; f''(x) = -0.6x$

(2 marks)

Let L_2 represent a lower estimate of the area of the shaded region using two rectangles of equal width.

(d) (i) Which *one* of the following statements could be used to justify that U_2 is a better approximation for the area of the shaded region than L_2?

Tick the appropriate box to indicate your answer.

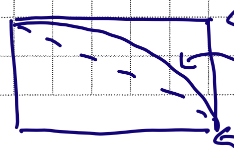
- [ ] $f(x) > 0$ for $0 \le x \le 1$
- [ ] $f'(x) < 0$ for $0 \le x \le 1$
- [x] $f''(x) \le 0$ for $0 \le x \le 1$

(1 mark)

(ii) Justify your answer.

as $f''(x) \le 0$ for $0 \le x \le 1$ the $f(x)$ is concave for $0 \le x \le 1$

$\therefore$



upper estimate

f(x) is closer to the upper than the lower

lower estimate

(1 mark)

(e) Using an algebraic approach, determine the *exact* area of the shaded region shown in Figure 1.

$\int_0^1 -\frac{1}{10}x^3 - 2x + 20 \, dx$

$= \left[-\frac{1}{10} \times \frac{x^4}{4} - \frac{2x^2}{2} + 20x \right]_0^1$

$= \left(-\frac{1}{40} - 1 + 20 \right) - (0)$

$= 18\frac{39}{40}$

(3 marks)

PLEASE TURN OVER

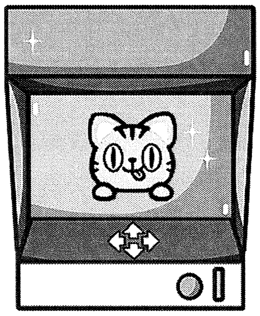
Question 4 (9 marks)

Cat Eyes is an arcade game where players try to guess correctly which direction an animated cat will look. The cat can look in four possible directions:

up, **down**, **left**, or **right**.

After a player has guessed, the cat looks in one of these directions.

A player's guess is correct if it matches the same direction that the cat looks. A game consists of *five* independent guesses.



Source: adapted from © Guiseppe Ramos © alashi | istockphoto.com

(a) A player makes the following assumptions when guessing:

- the cat looks in one of the four directions at random
- the probability that the cat looks in each of these directions is equally likely.

Let X represent the number of correct guesses when playing a game. When using the assumptions stated above, X can be modelled using a binomial distribution.

(i) Show that $E(X) = 1.25$.

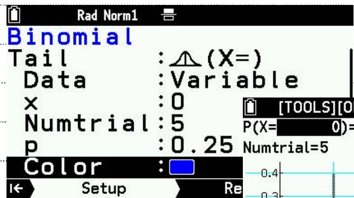
$X \sim Bin(5, 0.25)$

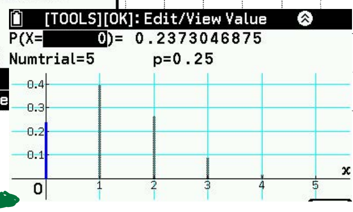
$E(X) = 5 \times 0.25 = 1.25$

(1 mark)

(ii) Determine the probability of zero correct guesses in a game.

$Pr(x = 0) = 0.237$





- Distribution App
- Binomial
- Enter data
- press Next →

- Press to hide the tabs

(1 mark)

(iii) A player *wins a prize* if they make more than k correct guesses in a game.

Using the binomial distribution, the probability of *winning a prize* is calculated to be 0.1035 (correct to four significant figures).

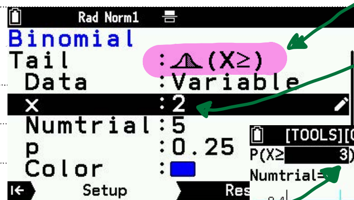
Determine the value of k.

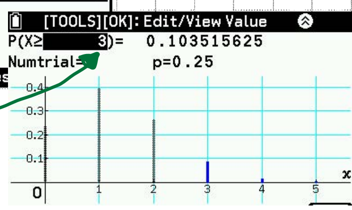
change the 'tail'

$Pr(x \geq 3) = 0.1035$

$\therefore k = 2$

This was my first guess





Type in 2, 3, ... until you see '0.1035'

(2 marks)

(b) After playing many games, the player suspects that the probability of *winning a prize* is less than anticipated (i.e. less than the probability of *winning a prize* when calculated using the binomial distribution).

To test their suspicions, they play 1000 games in which they *win a prize* 80 times.

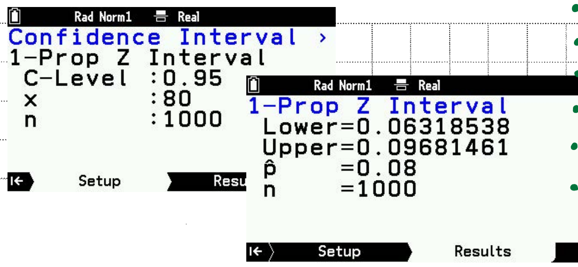
(i) Determine the sample proportion of games in which they *won a prize*.

$\hat{p} = \frac{80}{1000}$

(1 mark)

(ii) Construct a 95% confidence interval for the population proportion of games which *win a prize*.

$0.0632 \leq p \leq 0.0968$



- Statistics App
- Press Next →|
- Choose Conf. Int
- Choose 1- Prop Z
- Enter data
- Press Next →|

(2 marks)

(iii) (1) Does the confidence interval calculated in part (b)(ii) support the player's suspicions with 95% confidence?

Tick the appropriate box to indicate your answer.

☑ Yes ☐ No

(1 mark)

(2) Tick the appropriate box to justify your answer.

☐ The upper bound of the confidence interval is less than 0.08.

☑ The upper bound of the confidence interval is less than 0.1035.

☐ The upper bound of the confidence interval is less than 0.25.

☐ The upper bound of the confidence interval is less than 0.95.

(1 mark)

PLEASE TURN OVER

Question 5 (7 marks)

Consider the function $f(x)$ for $0 \le x \le 3$. Figure 2 shows a graph of the derivative of $f(x)$, (i.e. $y = f'(x)$ with x-intercepts at $x=0$, $x=1$, and $x=2$).

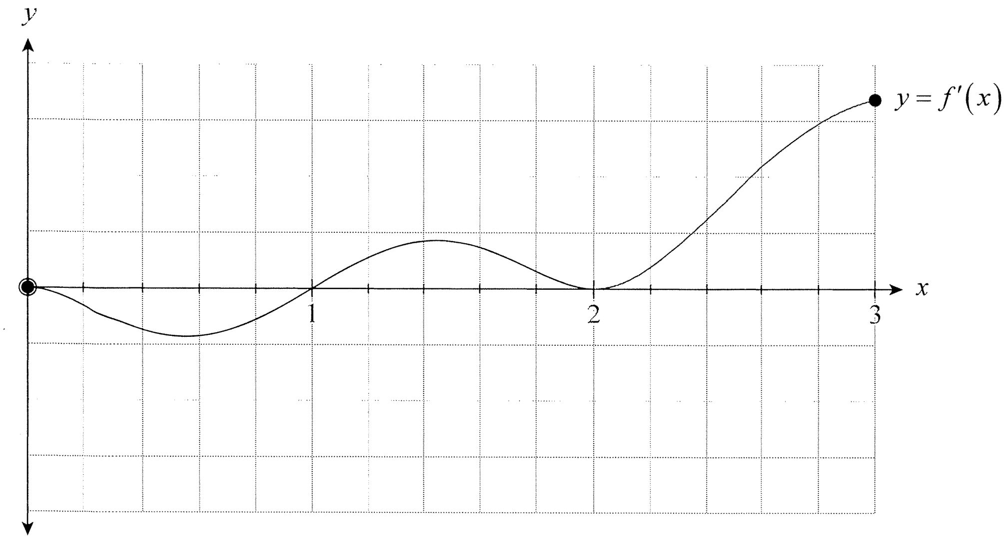
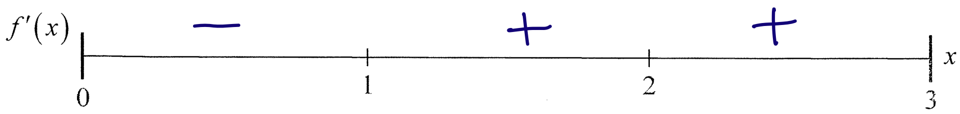


Figure 2

(a) Complete the sign diagram below for $f'(x)$.



(1 mark)

(b) State the value(s) of x where $f(x)$ is increasing.

$x \geq 1$

* depends on your definition of "increasing" as $f'(x) > 0$ or $f'(x) \geq 0$

or* $1 < x < 2$ and $2 < x < 3$

(1 mark)

(c) Which *one* statement is true? Tick the appropriate box to indicate your answer.

☐ $f(0) < f(1)$

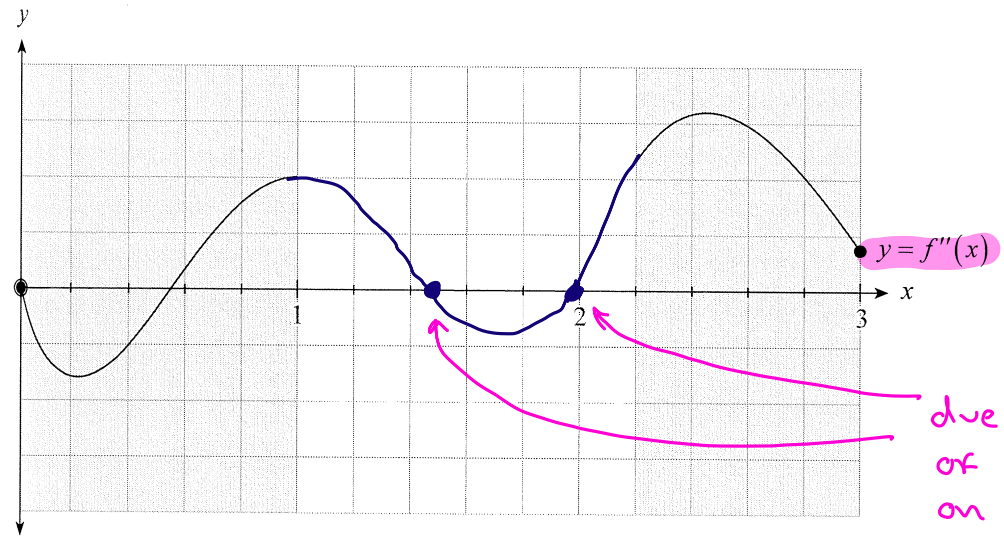
☐ $f(0) = f(1)$

☑ $f(0) > f(1)$ as $f(x)$ is decreasing from $x=0$ to $x=1$

(1 mark)

(d) Figure 3 shows an incomplete graph of $y = f''(x)$. ← can be thought of as the derivative of the graph above

On Figure 3, sketch a possible graph of $y = f''(x)$ for $1 \le x \le 2.2$ (i.e. in the non-shaded region).



due to location of stat. points on graph above

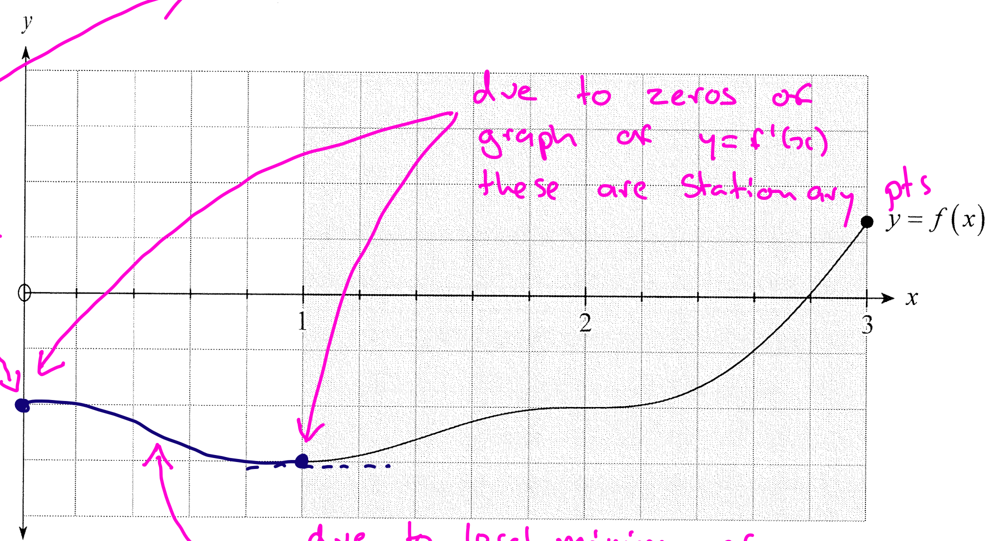
Figure 3

(2 marks)

(e) Figure 4 shows an incomplete graph of $y = f(x)$. ← now focus on fig. 2

On Figure 4, sketch a possible graph of $y = f(x)$ for $0 \le x \le 1$ (i.e. in the non-shaded region).

the height of this point cannot be precisely located



due to zeros of graph of $y=f'(x)$ these are stationary pts

due to local minima of graph of $y=f'(x)$ this is an inflection point

Figure 4

(2 marks)

PLEASE TURN OVER

Question 6 (8 marks)

Haz Finder is a wheeled autonomous robot that drives over land to investigate locations with hazardous substances, collecting samples of substances where possible.

Figure 5 shows a cross-section of a valley with a hazardous substance located at point O (the lowest point of the valley) and Haz Finder located at point H.

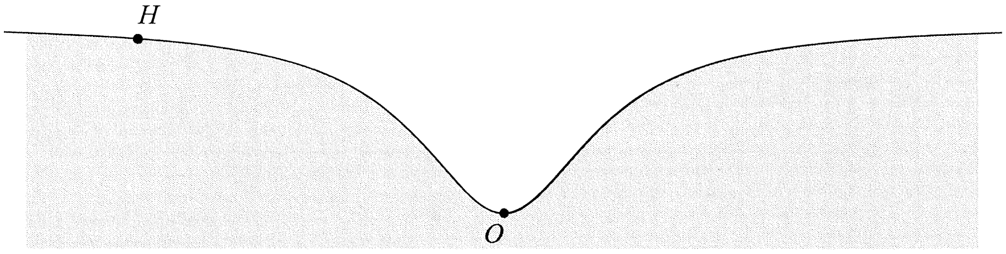


Figure 5

If the origin of the Cartesian plane is located at O, the valley can be modelled by the function

$$y = \frac{8x^2}{x^2 + 20} \text{ for } -20 \le x \le 20,$$

where x and y represent the horizontal and vertical displacement measured in metres from O respectively. A graph of this function is shown in Figure 6 where the x-coordinate of point H is $x = -15$.

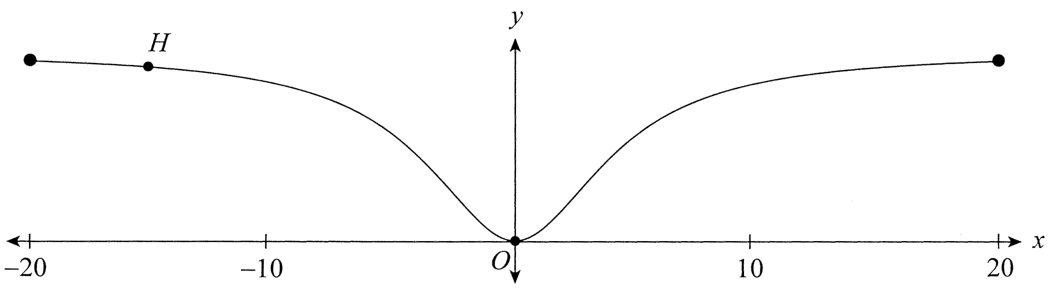


Figure 6

(a) Show that $\frac{dy}{dx} = \frac{320x}{(x^2+20)^2}$.

$$\frac{dy}{dx} = \frac{8 \times 2x(x^2+20) - 8x^2(2x)}{(x^2+20)^2}$$

$$= \frac{16x^3 + 320x - 16x^3}{(x^2+20)^2}$$

* to "show" the result given, this step should be shown.

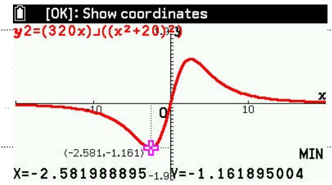
$$= \frac{320x}{(x^2+20)^2}$$

(2 marks)

(b) Haz Finder cannot drive on land that has a slope of less than -1.

Calculate the minimum value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ to confirm that Haz Finder cannot drive from H to O.

-1.16 which is less than -1



- Enter function in G+T
- Graph
- TOOLS ⊙
- Graph Solve ②
- minimum value ③
- Show coords (OK)

(1 mark)

As Haz Finder cannot drive from H to O to collect a sample, it will instead take a photograph of the hazardous substance at a point before the land gets too steep. Figure 7 shows the position of this point, A, with x-coordinate $x = a$.

The line through A and O is the tangent to $y = \dfrac{8x^2}{x^2+20}$ at $x = a$, where $a < 0$.

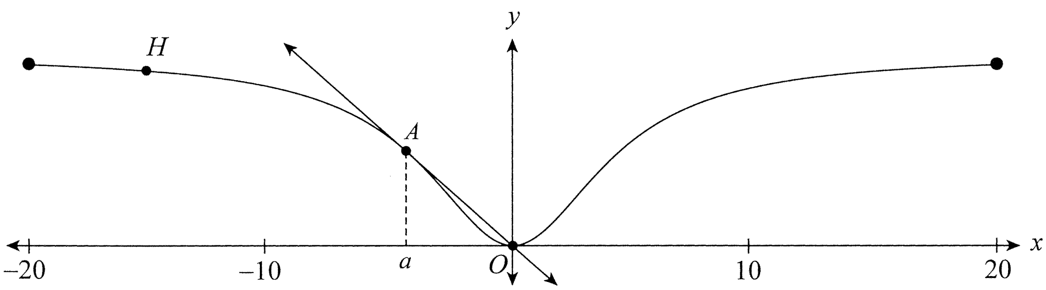


Figure 7

(c) Using an algebraic approach, determine the *exact* value of a.

$m = \dfrac{320a}{(a^2+20)^2}$ at $\left(a, \dfrac{8a^2}{a^2+20}\right)$

$\therefore$ tangent is

$320ax - (a^2+20)^2 y = 320a \times a - (a^2+20)^{\cancel{2}} \times \dfrac{8a^2}{\cancel{a^2+20}}$

$320ax - (a^2+20^2)y = 320a^2 - 8a^4 - 160a^2$

passes through origin when

$0 = 160a^2 - 8a^4$

$0 = 8a^2(20 - a^2)$

$\therefore a = -\sqrt{20}$ (as $a < 0$)

or

$\dfrac{8a^2}{a^2+20} = \dfrac{320a}{(a^2+20)^2} \times a + c$

$\therefore c = \dfrac{8a^2}{a^2+20} - \dfrac{320a^2}{(a^2+20)^2} = 0$

$8a^2(a^2+20) - 320a^2 = 0$

$8a^4 - 160a^2 = 0$

$8a^2(a^2 - 20) = 0$

$a = -\sqrt{20}$ (as $a < 0$)

(5 marks)

South Australian
Certificate of Education

Mathematical Methods
2025

Question booklet 2

- Questions 7 to 11 (50 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on pages 6, 10, and 18 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

2

Copy the information from your SACE label here

SEQ	FIGURES	CHECK LETTER	BIN

Graphics calculator

1. Brand ____________
   Model ____________
2. Brand ____________
   Model ____________

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of South Australia

Question 7 (6 marks)

The Total Daily Energy Expenditure (TDEE), measured in calories, is the amount of energy a person uses per day.

The TDEE of individuals, X, can be modelled by the probability density function

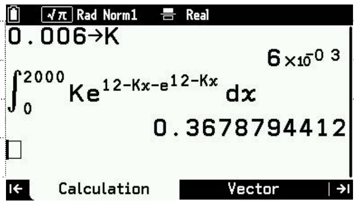
$$f(x) = ke^{\left(12-kx-e^{(12-kx)}\right)}$$

where $k = 0.006$, and $E(X) = 2100$ calories (correct to three significant figures).

(a) (i) The integral expression $\int_0^{2000} f(x)\,\mathrm{d}x$ can be used to approximate $\Pr(X \leq 2000)$.

Using this approximation, calculate $\Pr(X \leq 2000)$.

0.368



- Assigning 0.006 to k helps with error-free entry
- The "assign" arrow is SHIFT + (x / OPTN)
- Enter k using ALPHA + Left Bracket

(1 mark)

(ii) A recent article categorises individuals with a TDEE in the lowest 25% of the population as living a sedentary lifestyle.

If an individual in the study has a TDEE of 2000, are they considered to be living a sedentary lifestyle, according to the article's categorisation?

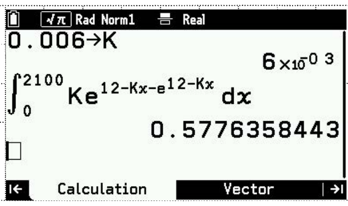
Justify your answer.

No, as 36.8% of the population have a TDEE of less than 2000 and so someone with a TDEE of 2000 is not in the lowest 25% of the population

(2 marks)

(b) Determine the probability that a randomly chosen individual will have a TDEE less than the population mean.

0.578



- Use arrow keys to edit the previous integral expression, changing the upper bound to 2100

(1 mark)

Fidgeting is a term used to describe minor movements, typically involving the hands and feet. Observations have led researchers to claim that:

'The population mean TDEE of individuals who fidget is more than 2109.5 calories.'

(c) Twenty-five randomly chosen individuals who fidget had their TDEE recorded, and a 99% confidence interval was calculated using the data.

(i) State the smallest possible value of the lower limit of the calculated confidence interval, given that it could be used to support the researchers' claim with 99% confidence.

Give your answer correct to the nearest calorie. ← lower bound is above 2109.5 to the nearest integer

2110

(1 mark)

Assume that your answer to part (c)(i) was the lower limit of the calculated confidence interval.

(ii) If the width of the calculated confidence interval was 168, determine the sample mean TDEE of the twenty-five randomly chosen individuals who fidget.

lower bound is 2110

upper bound is $2110 + 168 = 2278$

$\therefore \bar{x} = \dfrac{2110 + 2278}{2} = 2194$

midpoint of bounds

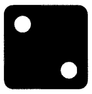
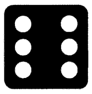
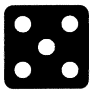
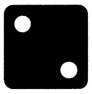
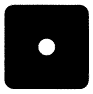
(1 mark)

Question 8 (11 marks)

(a) In a dice game, players repeatedly roll an unbiased six-sided dice until they roll a 1, upon which the game ends. Players receive 1 point for each roll of the dice.

For example, if a game resulted in the five rolls shown in Table 1, a player would receive a total of 5 points.

Table 1

Roll 1	*Roll 2*	*Roll 3*	*Roll 4*	*Roll 5*
				

(i) (1) Explain why the total number of points a player receives in a game *cannot* be modelled using a binomial distribution.

• not a fixed number of trials / rolls

(other reasoning also possible)

(1 mark)

(2) Explain why it is *not* possible for a player to receive a total of 0 points in a game.

The game requires at least one roll to obtain a '1' and hence the minimum number of points is 1

(1 mark)

(ii) The total number of points, X_6, a player receives in a game can be modelled by the probability mass function

$$\Pr(X_6 = x) = \left(\frac{5}{6}\right)^{x-1}\left(\frac{1}{6}\right)$$

where x is a positive integer.

(1) Determine the probability that a player receives a total of *exactly* 2 points.

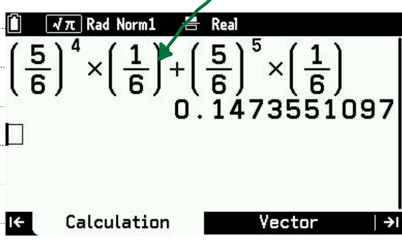
$\frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$

(1 mark)

(2) Determine the probability that a player receives a total of 5 or 6 points.

$\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^5\left(\frac{1}{6}\right)$

$= \frac{6875}{46656}$ $(= 0.147$ to 3 s.f.$)$



I used "copy + paste" to enter the second part of this

SHIFT (0) (<)(<)(<)(<)

(OK) (OK)

move to new location

SHIFT (.)

(2 marks)

It didn't really save time here but it does sometimes!

(b) If the six-sided dice from the game in part (a) was replaced with an unbiased n-sided dice $(n \geq 2)$ and all other conditions remained unchanged, the total number of points, X_n, a player receives in a game can be modelled by the probability mass function

$$\Pr(X_n = x) = \left(1 - \frac{1}{n}\right)^{x-1}\left(\frac{1}{n}\right)$$

where x is a positive integer.

(i) Consider a game where an unbiased ten-sided dice is used.

Determine the probability that a player receives a total of 2 points or fewer, when using this dice.

$$\left(\frac{9}{10}\right)^1\left(\frac{1}{10}\right) + \left(\frac{1}{10}\right)$$

$$= \frac{19}{100}$$

(2 marks)

(ii) (1) Consider a game where an unbiased n-sided dice is used. If the probability of a player receiving a total of 2 points or fewer is $\frac{39}{400}$ (i.e. $\Pr(X_n \leq 2) = \frac{39}{400}$), show that $39n^2 - 800n + 400 = 0$.

$$\left(1 - \frac{1}{n}\right)^1\left(\frac{1}{n}\right) + \left(\frac{1}{n}\right) = \frac{39}{400}$$

*

$$\therefore \frac{1}{n} - \frac{1}{n^2} + \frac{1}{n} = \frac{39}{400} \qquad \times 400n^2$$

$$\therefore 2 \times 400n - 400 = 39n^2$$

$$\therefore 39n^2 - 800n + 400 = 0$$

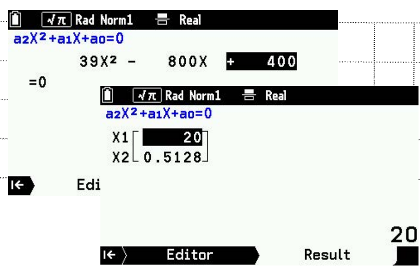
* many equivalent simplification processes could be used

(3 marks)

(2) Hence or otherwise, state the value(s) of n.

$n = 20$

(as n is integer)



• Equation App
• Polynomial
• Degree 2
• Enter data
• Press Next →

(1 mark)

Question 9 (8 marks)

Consider the function $f(x)=\sin x$, where $0\leq x\leq 2\pi$.

A graph of $y=f(x)$ is shown in Figure 8, along with two shaded regions bounded by $f(x)$ and the x-axis. Each region has an area of 2 square units. The regions are also identically shaped and symmetric about a vertical line through each of the stationary points.

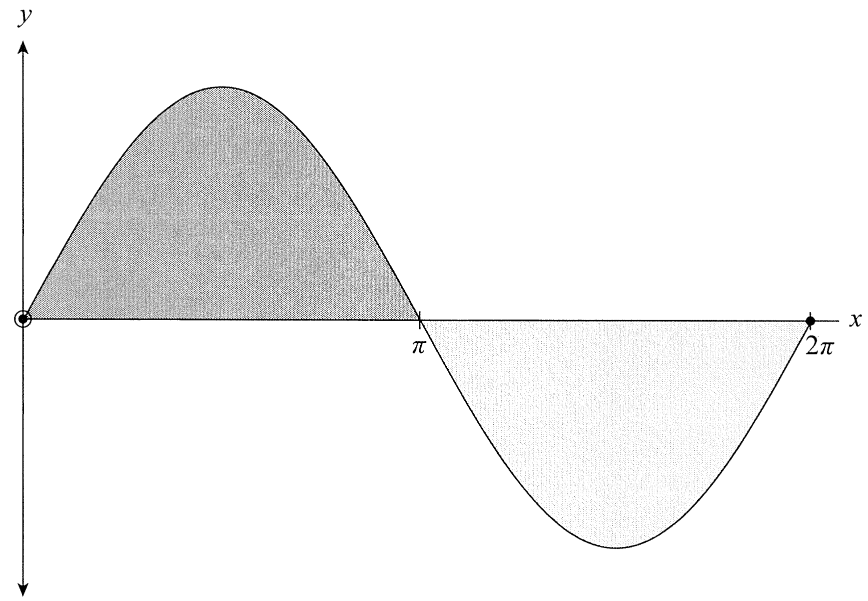


Figure 8

(a) State the value of $\int_0^{2\pi} f(x)\,dx$.

0 (as "positive area" and "negative area" will "cancel out")

(1 mark)

(b) State the value of a that maximises the value of $\int_0^{a} f(x)\,dx$ for $0\leq a\leq 2\pi$.

π (as a=π captures all the "positive area" and none of the "negative area")

(1 mark)

Question 9 continues on page 8

PLEASE TURN OVER

Recall that $f(x)=\sin x$, where $0 \le x \le 2\pi$.

Figure 9 shows the graph of $y=f(x)$, along with two linear functions, $y=g(x)$ and $y=h(x)$ both for $0 \le x \le 2\pi$. These linear functions have the following properties as shown in Figure 9:

- $g(x)$ intersects $f(x)$ at P_1, $(\pi, 0)$, and P_3
- $h(x)$ intersects $f(x)$ at P_1 and P_2 and is horizontal.

Various regions are bounded between the lines, the curve, the x-axis, and $x=2\pi$. Some of these regions are labelled as A, B, C, D, E, and F, as shown in Figure 9.

Note: the x-coordinates of P_1, P_2, and P_3 are p_1, p_2, and p_3 respectively.

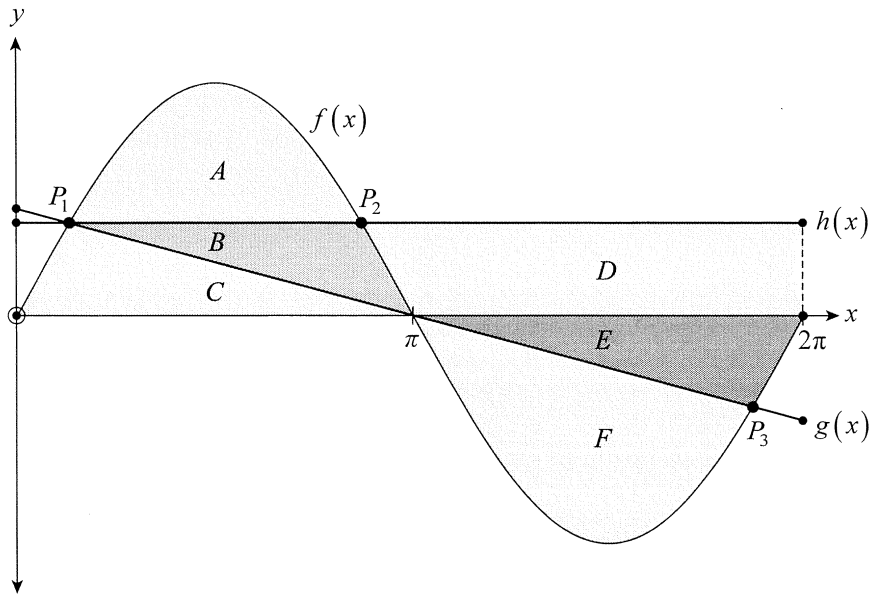


Figure 9

It is known that the area of:

- region A is 1 square unit
- region C is 0.566 square units (correct to three significant figures)
- region D is 1.20 square units (correct to three significant figures).

(c) State the area of region B.

$A + B + C = 2$

$\therefore \quad 1 + B + 0.566 = 2$

$\therefore \quad B = 0.434$

(1 mark)

(d) Evaluate $\displaystyle\int_{p_2}^{2\pi} \left(h(x) - f(x)\right) \mathrm{d}x$.

$= D + E + F = 1.20 + 2 = 3.2$

(1 mark)

(e) Write an expression using integrals and *two* of the functions given on page 8 that could be used to find the area of region F.

$$\int_{\pi}^{P_3} g(x) - f(x)\; dx$$

(2 marks)

(f) Which *one* statement is true? Tick the appropriate box to indicate your answer.

☐ $\int_0^{2\pi} \left(f(x) - h(x)\right) \mathrm{d}x > 0$

☐ $\int_0^{2\pi} \left(f(x) - h(x)\right) \mathrm{d}x = 0$

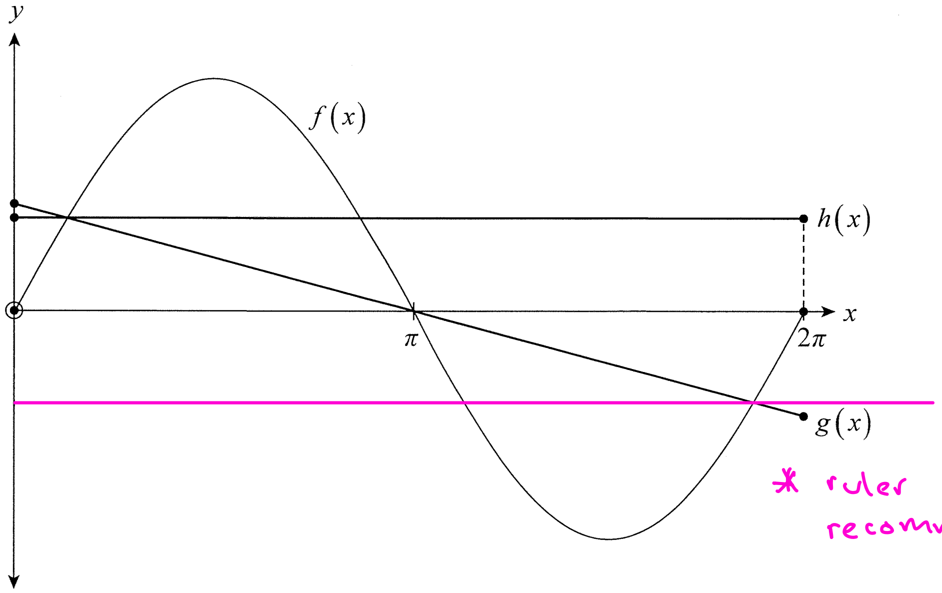
☑ $\int_0^{2\pi} \left(f(x) - h(x)\right) \mathrm{d}x < 0$

(1 mark)

(g) Let $n(x)$ be a horizontal line for $0 \le x \le 2\pi$ such that

$$\int_0^{2\pi} \left(f(x) - n(x)\right) \mathrm{d}x = -\int_0^{2\pi} \left(f(x) - h(x)\right) \mathrm{d}x.$$

On the axes in Figure 10, sketch a graph of $y = n(x)$.



* ruler use recommended

Figure 10

(1 mark)

Question 10 (12 marks)

Consider the function $f(x) = 100\ln(2x - \sin 2x)$, where $x > 0$ and $f'(x) \geq 0$ for all $x > 0$.
Let A_k represent the k^{th} closest stationary point of $f(x)$ to the origin, where k is a positive integer.
The graph of $y = f(x)$ is shown in Figure 11 with the three closest stationary points to the origin labelled A_1, A_2, and A_3.

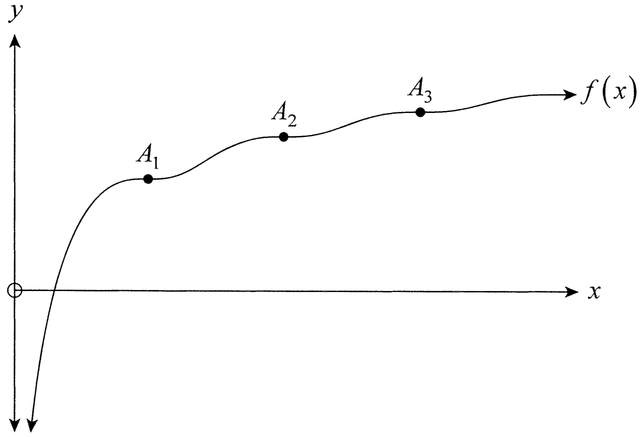


Figure 11

(a) Determine $f'(x)$.

$$f'(x) = 100 \times \frac{2 - (\cos 2x) \times 2}{2x - \sin 2x}$$

(2 marks)

(b) Using an algebraic approach, show that the *exact* x-coordinate of A_k is $x = \pi k$.

$f'(x) = 0$ when $2 - 2\cos 2x = 0$

$\therefore \cos 2x = 1$

$\therefore 2x = 0 + k2\pi$

$\therefore x = k\pi$

(2 marks)

PLEASE TURN OVER

(c) Figure 12 shows a section of the graph of the function $y = f(x)$, where $f(x) = 100\ln(2x - \sin 2x)$ located around consecutive stationary points A_k and A_{k+1} for some value of k.

The labelled distances between A_k and A_{k+1} are:

- d_k, the shortest distance
- v_k, the vertical distance
- h_k, the horizontal distance.

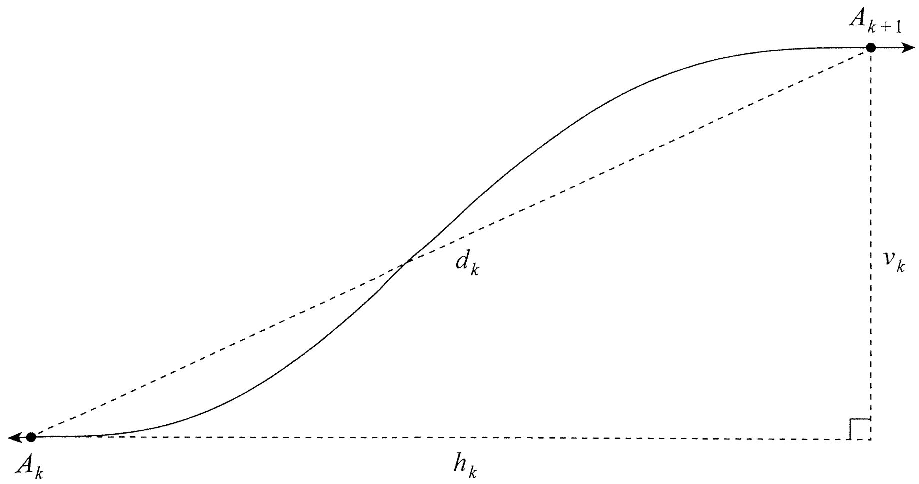


Figure 12

The following conjecture is made:

The vertical distance between the consecutive stationary points A_k and A_{k+1} is

$$v_k = 100\ln\left(\frac{k+1}{k}\right).$$

(i) Prove the given conjecture.

at $x = k\pi$, $\sin x = 0$ (for all k)

$V_K = 100\ln(2\pi(K+1)) - 100\ln(2\pi K)$

$= 100\ln\left(\frac{\cancel{2\pi}(K+1)}{\cancel{2\pi}K}\right)$

$= 100\ln\left(\frac{K+1}{K}\right)$

(4 marks)

(ii) (1) Hence, determine a formula for d_k, in terms of π and the variable k.

$$d_k = \sqrt{\left((k+1)\pi - k\pi\right)^2 + \left(100 \ln\left(\frac{k+1}{k}\right)\right)^2}$$

(1 mark)

(2) Hence, determine the smallest value of k such that $d_k < 3.50$.

Give evidence to support your answer.

$$\sqrt{\pi^2 + \left(100 \ln\left(\frac{k+1}{k}\right)\right)^2} < 3.5$$

$$\therefore \; 100 \ln\left(\frac{k+1}{k}\right) < 1.5429$$

$$\frac{k+1}{k} < 1.01555$$

$$\frac{1}{k} < 0.01555$$

$$k > \frac{1}{0.01555} \quad \left(\text{as } \frac{k}{k} = 1\right)$$

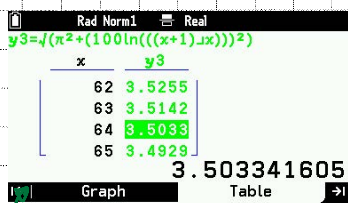
$$k > 64.32$$

$$\therefore \; k = 65$$

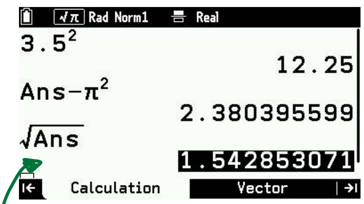
or

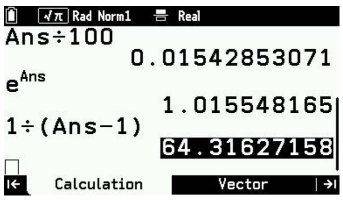
k	d_k
64	3.50334
65	3.49293

$\therefore \; k = 65$



(3 marks)





Answer command was very useful here for speed and accuracy

Call up 'Ans' by pressing ALPHA + FORMAT

- Enter the expression for d_k into G+T app
- Choose 'Display Table'
- Use TOOLS (…) and Set Table Domain ①
- Use arrows to search for the tipping point.

Question 11 (13 marks)

The family of functions known as *hyperbolic* functions are expressed using the following notation:

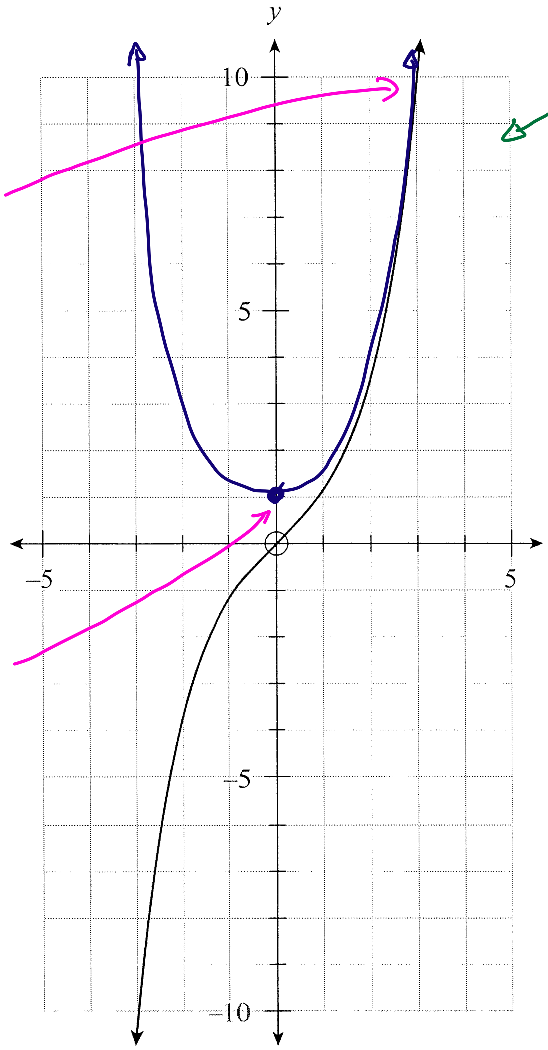
$$\sinh x = \frac{e^{x} - e^{-x}}{2} \qquad \cosh x = \frac{e^{x} + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

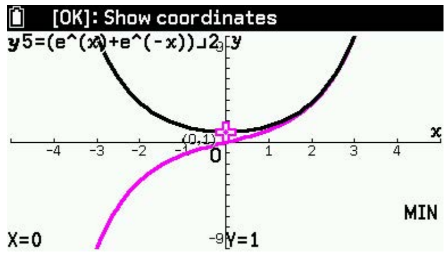
Figure 13 shows the graph of $y = \sinh x$.

• Enter the function into G+T
• Graph on the view window provided $(-5 \leq x \leq 5$ and $-10 \leq y \leq 10)$

Keep your sketch above / "within" the curve shown

plot the local minima for accuracy





• Use TOOLS (...)
• Graph Solve (2)
• Minimum value (2)
to locate (and plot) the critical value of (0, 1)

Figure 13

(a) On Figure 13, add a sketch of the graph of $y = \cosh x$.

(2 marks)

One reason for expressing *hyperbolic* functions using the notation on page 14 is that their derivatives follow a similar *(but not identical)* pattern to trigonometric functions. Two examples of the relationship between *hyperbolic* functions and their derivatives are shown in Table 2.

Table 2

Relationship 1	*Relationship 2*
If $y = \sinh x$, $\frac{dy}{dx} = \cosh x$	If $y = \cosh x$, $\frac{dy}{dx} = \sinh x$

(b) A third relationship is shown in Table 3.

Table 3

Relationship 3
If $y = \tanh x$, $\frac{dy}{dx} = \frac{1}{(\cosh x)^2}$

Show that Relationship 3 is valid.

You may assume that Relationship 1 and Relationship 2 are valid when constructing your answer, if required.

$$y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\therefore \frac{dy}{dx} = \frac{(e^x - -e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x + -e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$= \frac{e^{2x} + 2 + e^{-x} - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2}$$

$$= \frac{4}{(e^x + e^{-x})^2}$$

$$\frac{1}{(\cosh x)^2} = \frac{1}{\left(\frac{e^x + e^{-x}}{2}\right)^2}$$

$$= \frac{4}{(e^x + e^{-x})^2}$$

$\therefore$ Rel. 3 is valid

(3 marks)

Recall that *hyperbolic* functions are expressed using the following notation:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Additionally, recall the following relationships in Table 4:

Table 4

Relationship 1	*Relationship 2*	*Relationship 3*
If $y = \sinh x$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \cosh x$	If $y = \cosh x$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh x$	If $y = \tanh x$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{(\cosh x)^2}$

(c) Consider the function $f(x) = 13\tanh x + 10\ln(\cosh x)$.

(i) Show that $f''(x) = \dfrac{10\cosh x - 26\sinh x}{(\cosh x)^3}$.

$$f'(x) = 13 \times \frac{1}{(\cosh x)^2} + 10 \times \frac{\sinh x}{\cosh x}$$ tanh x

$$f''(x) = 13 \times -2(\cosh x)^{-3} \times \sinh x + 10 \times \frac{1}{(\cosh x)^2}$$

$$= \frac{-26\sinh x}{(\cosh x)^3} + \frac{10}{(\cosh x)^2}$$

$$= \frac{10\cosh x - 26\sinh x}{(\cosh x)^3}$$

(4 marks)

(ii) The function $f(x)$ has one point of inflection.

Using an algebraic approach and your answer to part (c)(i), determine the *exact* x-coordinate of the point of inflection.

Express your answer in the form $x = \ln\left(\frac{a}{b}\right)$, where a and b are integers.

$$f''(x) = 0 \Rightarrow 10\cosh x - 26\sinh x = 0$$

$$\therefore \quad 10\left(\frac{e^x + e^{-x}}{2}\right) - 26\left(\frac{e^x - e^{-x}}{2}\right) = 0$$

$$5e^x - 13e^x = -5e^{-x} - 13e^{-x}$$

$$-8e^x = \frac{-18}{e^x}$$

$$e^{2x} = \frac{9}{4}$$

$$2x = \ln\left(\frac{9}{4}\right)$$

$$x = \frac{1}{2}\ln\left(\frac{9}{4}\right) = \ln\left(\frac{9}{4}\right)^{1/2}$$

$$= \ln\frac{3}{2}$$

(4 marks)

MATHEMATICAL METHODS FORMULA SHEET

Properties of derivatives

$$\frac{d}{dx}\left(f(x)\,g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

$$\frac{d}{dx} f\left(g(x)\right) = f'\left(g(x)\right)g'(x)$$

Quadratic equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Discrete random variables

The mean or expected value of a discrete random variable is:

$$\mu_X = \sum xp(x),$$

where $p(x)$ is the probability function for achieving result x.

The standard deviation of a discrete random variable is:

$$\sigma_X = \sqrt{\sum [x - \mu_X]^2\, p(x)},$$

where μ_X is the expected value and $p(x)$ is the probability function for achieving result x.

Bernoulli distribution

The mean of the Bernoulli distribution is p, and the standard deviation is:

$$\sqrt{p(1-p)}.$$

Binomial distribution

The mean of the binomial distribution is np, and the standard deviation is:

$$\sqrt{np(1-p)},$$

where p is the probability of success in a single Bernoulli trial and n is the number of trials.

The probability of k successes from n trials is:

$$\Pr(X = k) = C_k^n p^k (1-p)^{n-k},$$

where p is the probability of success in the single Bernoulli trial.

Population proportions

The sample proportion is $\hat{p} = \dfrac{X}{n}$,

where a sample of size n is chosen, and X is the number of elements with a given characteristic.

The distribution of a sample proportion has a mean of p and a standard deviation of

$$\sqrt{\frac{p(1-p)}{n}}.$$

The upper and lower limits of a confidence interval for the population proportion are:

$$\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},$$

where the value of z is determined by the confidence level required.

Continuous random variables

The mean or expected value of a continuous random variable is:

$$\mu_X = \int_{-\infty}^{\infty} x f(x)\,dx,$$

where $f(x)$ is the probability density function.

The standard deviation of a continuous random variable is:

$$\sigma_X = \sqrt{\int_{-\infty}^{\infty} [x - \mu_X]^2 f(x)\,dx},$$

where $f(x)$ is the probability density function.

Normal distributions

The probability density function for the normal distribution with mean μ and standard deviation σ is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

All normal distributions can be transformed to the standard normal distribution with $\mu = 0$ and $\sigma = 1$ by:

$$Z = \frac{X - \mu}{\sigma}.$$

Sampling and confidence intervals

If $\bar{x}$ is the sample mean of a sufficiently large sample, and σ is the population standard deviation, then the upper and lower limits of the confidence interval for the population mean are:

$$\bar{x} - z\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + z\frac{\sigma}{\sqrt{n}},$$

where the value of z is determined by the confidence level required.